Suggested to me by Tom O’Neil:

Let \( \{x_n\}_{n=0}^\infty \) be the sequence defined recursively by

\[
x_0 = a, \quad x_1 = b, \quad x_{n+1} = x_n - 1 + \frac{(2n - 1)x_n}{2n}, \quad \text{for } n \geq 1.
\]

Calculate, in terms of \( a \) and \( b \), the value of \( \lim_{n \to \infty} x_n \).

\[\text{Solution:}\]

The sequence converges to \((1 - e^{-1/2})a + e^{-1/2}b\).

We rewrite the recursion relation as \(2n(x_{n+1} - x_n) = -(x_n - x_{n-1})\), which suggests we introduce \(y_n = (x_{n+1} - x_n)\). Then we have the new sequence:

\[
y_0 = b - a, \quad y_n = -\frac{1}{2n}y_{n-1} \quad \text{for } n \geq 1.
\]

Then we calculate \(y_n = -\frac{1}{2n}y_{n-1} = \left(\frac{-1}{2n}\right)\left(\frac{-1}{2(n-1)}\right)\left(\frac{-1}{2(n-2)}\right) \cdots \left(\frac{-1}{2(1)}\right)y_0 = \left(\frac{-1}{2}\right)^n \frac{1}{n!}y_0\). Now

\[
x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \ldots + (x_1 - x_0) = \sum_{i=0}^{n} y_i = y_0 \sum_{i=0}^{n} \left(\frac{-1}{2}\right)^i \frac{1}{i!}
\]

So

\[
x_{n+1} = a + (b - a) \sum_{i=0}^{n} \left(\frac{-1}{2}\right)^i \frac{1}{i!}.
\]

Now take a limit on both sides and use the fact that \(\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x\) to reach the answer given above.