Cal Poly Department of Mathematics

Puzzle of the Week

Nov 6 - 12, 2014

Suggested to me by Tom O'Neil:

Let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined recursively by

$$x_0 = a,\ x_1 = b, \quad x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n},\ \text{for } n \geq 1.$$

Calculate, in terms of a and b, the value of $\lim_{n\to\infty} x_n$.

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The sequence converges to $(1 - e^{-1/2})a + e^{-1/2}b$.

We rewrite the recursion relation as $2n(x_{n+1} - x_n) = -(x_n - x_{n-1})$, which suggests we introduce $y_n = (x_{n+1} - x_n)$. Then we have the new sequence:

$$y_0 = b - a, \quad y_n = -\frac{1}{2n} y_{n-1} \text{ for } n \geq 1.$$

Then we calculate $y_n = -\frac{1}{2n} y_{n-1} = \left(\frac{-1}{2n}\right)\left(\frac{-1}{2(n-1)}\right)\left(\frac{-1}{2(n-2)}\right) \cdot \ldots \cdot \left(\frac{-1}{2(1)}\right) y_0 = \left(\frac{-1}{2}\right)^n \frac{1}{n!} y_0$. Now

$$x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \ldots + (x_1 - x_0) = \sum_{i=0}^{n} y_i = y_0 \sum_{i=0}^{n} \left(\frac{-1}{2}\right)^i \frac{1}{i!}$$

So

$$x_{n+1} = a + (b - a) \sum_{i=0}^{n} \left(\frac{-1}{2}\right)^i \frac{1}{i!}.$$

Now take a limit on both sides and use the fact that $\sum_{i=0}^{\infty} x^i \frac{1}{i!} = e^x$ to reach the answer given above.