A sequence of positive integers \( \{a_n\}_{n=1}^\infty \) satisfies
\[
a_{n+3} = a_{n+2}(a_{n+1} + a_n), \quad n = 1, 2, 3, \ldots
\]
If \( a_6 = 8820 \), determine the possible values of \( a_1, a_2, a_3, a_7, \) and \( a_8 \).

**Solution:** There are two possibilities: \( (a_1, a_2, a_3) = (2, 2, 7) \) or \( (29, 6, 1) \).

We first calculate that \( 8820 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \) and, using the recurrence relation repeatedly, we get \( a_6 = a_3^2(a_3 + a_2)(a_2 + a_1)(a_2 + a_1 + 1) \). We set \( x = a_3, y = a_3 + a_2, z = a_2 + a_1 \). Then we are looking for positive integer solutions to
\[
2^2 \cdot 3^2 \cdot 5 \cdot 7^2 = x^2 \cdot y \cdot z \cdot (z + 1), \quad x < y < x + z. \tag{1}
\]

We notice that one of \( z \) or \( z + 1 \) is even, so \( x \) cannot be, which leaves only four possibilities for \( x^2 : 1, 3^2, 7^2, 3^2 \cdot 7^2 \).

If \( x^2 = 1 \): Then (1) implies that \( 8820 = y \cdot z \cdot (z + 1) < z(z + 1)^2 \). It follows that \( z > 20 \). The only pair of divisors \( z, z + 1 \) to 8820 with \( z > 20 \) is \( z = 5 \cdot 7 = 35 \) and \( z + 1 = 2^2 \cdot 3^2 = 36 \). This leads to the solution \( x = 1, y = 7, z = 35 \), which gives \( a_1 = 29, a_2 = 6, a_3 = 1 \).

If \( x^2 = 3^2 \): Then (1) implies that \( \frac{8820}{9} < z(z + 1)(z + 3) \). This implies \( z > 8 \). But there are no pairs of divisors \( z, z + 1 \) to \( \frac{8820}{9} = 2^2 \cdot 5 \cdot 7^2 \) with \( z > 8 \).

If \( x^2 = 7^2 \): Then (1) becomes \( 2^2 \cdot 3^2 \cdot 5 = yz(z + 1) \) and \( 7 < y < z + 7 \). Also \( z \) and \( z + 1 \) must both divide \( 2^2 \cdot 3^2 \cdot 5 \). A quick check and we find the only possibility is \( (x, y, z) = (7, 9, 4) \), which leads to \( a_1 = 2, a_2 = 2, a_3 = 7 \).

If \( x^2 = 3^2 \cdot 7^2 \): Then (1) becomes \( 2^2 \cdot 5 = yz(z + 1) \), \( 21 < y < z + 21 \). So \( 20 > y(y - 21)(y - 20) \) which gives \( y < 1 \), a contradiction.