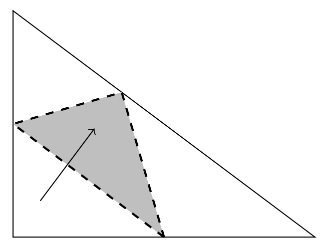
Cal Poly Department of Mathematics

Puzzle of the Week

May 22 - 28, 2014

A sheet of paper in the shape of a "3-4-5" right triangle has its right-angled corner folded up until it touches the opposite side. Correct to three decimal places, what is the minimum area of the triangle which is folded up?



Solutions should be submitted to Morgan Sherman:

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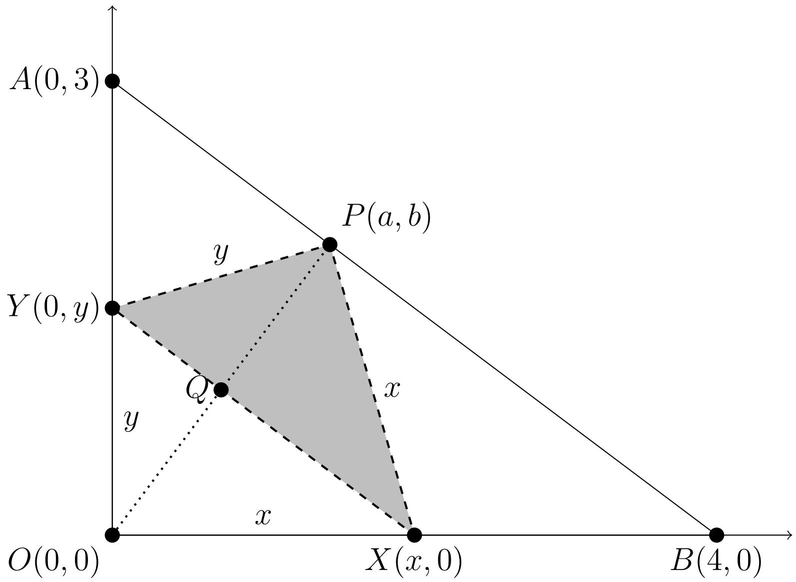
before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> The minimum area is about 1.459.

Set up coordinates and label the triangle as shown:



Now, the fact that P lies on AB implies that

$$3a + 4b = 12. \tag{1}$$

That the triangle PXY is the reflection of OXY across XY implies that the line OP is perpendicular to XY, which means that

$$ax = by. \tag{2}$$

Combining (1) and (2) we find

$$a = \frac{12y}{4x+3y}, \quad b = \frac{12x}{4x+3y}. \tag{3}$$

Let Q denote the intersection of OP and XY (as shown in the diagram). Since $\triangle QYP \sim \triangle PYX$, and find that $|QP|/|YP| = |PX|/|XY|$, which implies that

$$|QP| = \frac{|PX| \cdot |YP|}{|XY|} = \frac{xy}{\sqrt{x^2+y^2}}.$$

On the other hand $|OP| = 2|QP| = \sqrt{a^2+b^2}$, so by (3) we have

$$|QP| = \frac{6\sqrt{x^2+y^2}}{4x+3y}.$$

Thus we find

$$6(x^2+y^2) = xy(4x+3y). \tag{4}$$

So we must minimize the function $A = \frac{1}{2}xy$ subject to the constraint (4). Using Lagrange multipliers and a computer algebra system (I used *Sage*) we find the minimum occurs at approximately $x = 1.79138$ and $y = 1.62946$, leading to the solution above.

Addendum: Matt Carlton finds an exact answer by calculating the area in question in terms of the angle $\alpha = \angle POB$. Minimizing this function the exact area can be computed as

$$\text{Area} = \frac{36}{25} \csc^3 \left(\frac{\pi}{3} + \frac{2}{3} \arctan \frac{4}{3} \right) = 1.45948898537\ldots$$

See the Puzzle of the Week website for a link to his solution.