Cal Poly Department of Mathematics

Puzzle of the Week

May 15 - 21, 2014

Relayed to me by Bob Wolf:

For nonnegative integers n, m, let $a_n = 2^n - 1$ and let $b_m = 2^m + 1$. Is there any a_n, other than $a_n = 1, 3$, for which a_n divides b_m, for some m?

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>Solution:</u> Bob tells me he found this in an old *Mathematics Magazine*, which reminds me that publication is an excellent source of fun math problems (and of course lots of other math related news and articles).

The answer is that if a_n is not 1 or 3, then it will not divide any b_m for any m.

First, it is not too hard to verify that if $m \leq n$, then we must have $n = 1$ or $n = 2$, which give $a_n = 1$ or 3.

Now assume $m > n$. We wish to show $a_n \nmid b_m$. The crucial observation is that if we write $m = n+k$, then

$$b_m = 2^m + 1 = 2^{n+k} + 1 = 2^k(2^n - 1) + 2^k + 1 \equiv 2^k + 1 \pmod{a_n}$$

It follows that if $m = an + k$, with $0 \leq k < n$, then

$$b_m \equiv 2^k + 1 \pmod{a_n}$$

Since $k < n$ this will not be zero, unless $n = 1$ (and k is any number) or $n = 2$ (and $k = 1$) as one can check.