

Cal Poly Department of Mathematics

Puzzle of the Week

May 8 - 14, 2014

From Kent Morrison:

Determine the positive integers that can be expressed as the sum of positive integers $x + y + z$ such that $x < y < z$, x divides y , and y divides z .

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

<http://www.calpoly.edu/~sherman1/puzzleoftheweek>

Solution: Every number *except* 1, 2, 3, 4, 5, 6, 8, 12, 24 can be so expressed.

Let E be the set of all positive integers which are expressible as in the problem. Evidently the smallest number in E is $1 + 2 + 4 = 7$, so each of 1, 2, 3, 4, 5, and 6 are not in E . The next largest number would be $1 + 2 + 6 = 9$, so 8 is also not in E . Suppose $x + y + z = 12$. We may assume $x = 1$ since otherwise we could divide both sides by x and get a factor of 12 in E , which we know is impossible. Write $z = ay$ for $a > 1$. Then our equation becomes $1 + y + ay = 12$, which implies that $11 = y(a + 1)$ which is a contradiction since 11 is prime. Similar reasoning shows that 24 is not in E (since again we could assume $x = 1$, so as before we would arrive at the equation $23 = y(a + 1)$).

It remains to show all other numbers are in E . First every odd number $n \geq 7$ is in E since $n = 1 + 2 + (n - 3)$ (note that $2 \mid (n - 3)$ when n is odd). Next note that if n is in E then so is every multiple of n (since we can just multiply both sides of the equation $x + y + z = n$ to obtain our multiple of n). Now every positive integer is uniquely expressible as $n = 2^k \ell$ where ℓ is an odd integer. So if $\ell \geq 7$ then $n \in E$ by the above.

If $\ell = 1$ then n is a power of 2. By experimentation we find $16 = 1 + 5 + 10$, so $16 \in E$ and so are all larger powers of 2. And we have already ruled out the numbers 1, 2, 4, 8.

If $\ell = 3$, note that we have already ruled out $1 \times 3, 2 \times 3, 4 \times 3, 8 \times 3$. Next is 48, which is a multiple of 16, so $48 \in E$ and so are all multiples of it.

Finally if $\ell = 5$, then we note that $1 + 3 + 6 = 10 \in E$, so all multiples of this are in E . Other multiples of 5 are odd, which we have taken care of above.