Cal Poly Department of Mathematics

Puzzle of the Week

April 24 - ~~30~~ May 7, 2014

Shown to me by Joe Borzellino:

Everyone who has gone through Calculus III has seen that the "alternating p-series" follows:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \quad \begin{cases} \text{converges absolutely} & \text{for } p > 1 \\ \text{converges conditionally} & \text{for } 0 < p \leq 1 \\ \text{diverges} & \text{for } p \leq 0 \end{cases}$$

The problem for this week: for which p is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{n^p}$$

absolutely convergent, conditionally convergent, divergent? [Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x]

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: The series converges absolutely for $p > 1$, diverges for $p \leq 1/2$, and converges conditionally for $1/2 < p \leq 1$.

First, for $p > 1$ we get absolute convergence since the series $\sum \frac{1}{n^p}$ is the well known p-series, which converges for $p > 1$ (and diverges for $p \leq 1$).

Now the original series (pseudo-) "alternates" in that it changes sign at every perfect square:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = \underbrace{\frac{-1}{1^p} + \frac{-1}{2^p} + \frac{-1}{3^p}}_{1 \le n < 4} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \ldots + \frac{1}{8^p}}_{4 \le n < 9} + \underbrace{\frac{-1}{9^p} + \ldots + \frac{-1}{15^p}}_{9 \le n < 16} + \ldots$$

Then it is not hard to see the original series converges if and only if the series

$$\sum_{n=1}^{\infty} (-1)^n a_n, \quad \text{where } a_n = \frac{1}{(n^2)^p} + \frac{1}{(n^2+1)^p} + \ldots + \frac{1}{(n^2+2n)^p}$$

converges. (Note that in general one has to be careful about re-grouping terms in an infinite sum; however here it is valid.) So we would like to examine the terms a_n. First note that

$$a_n > \frac{1}{(n+1)^{2p}} + \frac{1}{(n+1)^{2p}} + \ldots + \frac{1}{(n+1)^{2p}} = \frac{2n+1}{(n+1)^{2p}} > \frac{1}{(n+1)^{2p-1}}$$

so we see that the series will diverge if $2p - 1 \le 0$, since the a_n will not even approach 0.

Using the binomial series $(1+x)^\ell = \sum_i \binom{\ell}{i} x^i$ for $|x| < 1$ we compute

$$\begin{aligned}
a_n &= \sum_{k=0}^{2n} \frac{1}{(n^2+k)^p} \\
&= \frac{1}{n^{2p}} \sum_{k=0}^{2n} \left(1 + \frac{k}{n^2}\right)^{-p} \\
&= \frac{1}{n^{2p}} \sum_{k=0}^{2n} \left(1 - p\frac{k}{n^2} + \ldots\right) \\
&= \frac{1}{n^{2p}} \left((2n+1) - p\frac{(2n)(2n+1)}{2n^2} + \ldots\right) \\
&= \frac{2}{n^{2p-1}} + O\left(\frac{1}{n^{2p}}\right)
\end{aligned}$$

So if $2p - 1 > 0$ then the a_n will go to zero and will eventually decrease. By the *alternating series test*, the series converges if $p > 1/2$.