Cal Poly Department of Mathematics

Puzzle of the Week

Feb 27 - Mar 5, 2014

What is the units digit (i.e. rightmost digit) of the number

$$\left\lfloor \frac{10^{4028000}}{10^{2014}-7} \right\rfloor$$

(here $\lfloor x \rfloor$ is the greatest integer $\leq x$)?

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

Solution: The ones digit is 3. This problem is a variation of a 1986 Putnam problem.

Let $N = 10^{2014}$ and rewrite the number inside $\lfloor \cdot \rfloor$ using a geometric series:

$$\begin{aligned}
\frac{N^{2000}}{N-7} = \frac{N^{2000}}{N}\frac{1}{1-(7/N)} &= N^{1999}\sum_{i=0}^{\infty}\left(\frac{7}{N}\right)^i \\
&= \left(\sum_{i=0}^{1998} N^{1999-i}7^i\right) + \left(7^{1999}\right) + \left(N^{1999}\sum_{i=2000}^{\infty}\left(\frac{7}{N}\right)^i\right) \\
&= (A) + (B) + (C)
\end{aligned}$$

Now A and B are both integers and A is divisible by 10. Note that C is smaller than 1 since:

$$C = N^{1999}\sum_{i=2000}^{\infty}\left(\frac{7}{N}\right)^i = N^{1999}\frac{(7/N)^{2000}}{1-(7/N)} = \frac{7^{2000}}{N-7} = \frac{7^{2000}}{10^{2014}-7} < 1$$

Therefore $\left\lfloor \frac{10^{4028000}}{10^{2014}-7} \right\rfloor = A + B$ and the ones digit will be the same as the ones digit of $B = 7^{1999}$. Finally $7^2 = 49 \equiv -1 \pmod{10}$ so we find

$$7^{1999} = 7^{2\cdot 999+1} \equiv (-1)^{999}\cdot 7 = -7 \equiv 3 \pmod{10}$$