For every $r > 1$ calculate the value of

$$\sum_{n=0}^{\infty} \frac{2^n}{r^{2^n} + 1} = \frac{1}{r+1} + \frac{2}{r^2 + 1} + \frac{4}{r^4 + 1} + \frac{8}{r^8 + 1} + \cdots$$

**Solution:** The series sums to $1/(r - 1)$.

The trick here is to rewrite each term as a geometric series, and then gather all terms with common powers of $r$:

$$\sum_{n=0}^{\infty} \frac{2^n}{r^{2^n} + 1} = \sum_{n=0}^{\infty} \frac{2^n}{r^{2^n} + 1} = \sum_{m,n \geq 0} \frac{2^n (-1)^m r^{-2n} m^n}{r^{2n} + 1}$$

Now write $m + 1 = 2^k (2\ell + 1)$. Then the power of $r^{-1}$ in the general term is $2^{k+n}(2\ell + 1)$. So two such terms have a common power of $r$ if and only if they share the same $\ell$ and the same value for $n + k$. Also note that $(-1)^m = -1 \iff k > 0$. Therefore we can rewrite the series:

$$\sum_{m,n \geq 0} (-1)^m 2^n r^{-2n}(m+1) = \sum_{k',\ell \geq 0} \left[ \sum_{n=0}^{k'} (-1)^m 2^n \right] r^{-2k'(2\ell+1)} = \sum_{k,\ell \geq 0} \left[ - \left( \sum_{n=0}^{k-1} 2^n \right) + 2^k \right] r^{-2k(2\ell+1)}$$

$$= \sum_{k,\ell \geq 0} \left[ -\frac{2^k - 1}{2 - 1} + 2^k \right] r^{-2k(2\ell+1)} = \sum_{k,\ell \geq 0} r^{-2k(2\ell+1)} = \sum_{i=1}^{\infty} r^{-i} = \frac{r^{-1}}{1 - r^{-1}}$$