# Cal Poly Department of Mathematics

# Puzzle of the Week

Jan 30-Feb 5, 2014

Suppose for a given $n$ that $2^n$ and $5^n$ both begin with the same digit. What are the possible values of that digit (i.e. which of the numbers $1, 2, 3, \ldots, 9$ could this digit be)?

*Solutions should be submitted to Morgan Sherman:*


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*before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.*

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The only possible leading digit for a postive power $n > 0$ can only be a 3 and this occurs (for example for $n = 5$ we have $2^n = 32$ and $5^n = 3125$); if $n = 0$ then we have a leading digit of 1.

Suppose $n > 0$ and that $2^n$ and $5^n$ have the same leading digit. Write

$$2^n = a \cdot 10^k, \quad 5^n = b \cdot 10^l$$

for real numbers $1 < a, b < 10$. Then

$$10^n = ab \cdot 10^{k+l}$$

and it follows that $ab$ is an integer power of 10. Then since $1 < ab < 100$ we have that $ab = 10$. Now write $a = d + \epsilon_1,\ b = d + \epsilon_2$ for some $d = 1, 2, 3, \ldots, 9$, and $0 < \epsilon_i < 1$. Then by considering each of the possible values for $d$ in the equation

$$d^2 + d(\epsilon_1 + \epsilon_2) + \epsilon_1\epsilon_2 = 10$$

we find: if $d \geq 4$ then the left-hand-side is $\geq 16$, a contradiction; if $d \leq 2$ then the left-hand-side is $\leq 7$, another contradiction. The only possiblility left is $d = 3$.