In the triangle $ABC$ let $M$ denote the midpoint of $BC$, let $O$ denote the circumcenter, and let $D$ denote the second point intersection of $AM$ with the circumcircle of $ABC$.

Suppose we are given that $\angle BAC = 45^\circ$ and that $|AM| = 2|MD|$. Find $\cos(\angle AOD)$.

**Solution:**

The cosine of $\angle AOD$ is $-\frac{1}{8}$.

Without loss of generality assume that the radius of the circumcircle is 1. An important observation is that since $\angle BAC = 45^\circ$ it follows that $\angle BOC = 90^\circ$. Therefore $|BC| = \sqrt{2}$ and $|BM| = |MC| = \frac{1}{\sqrt{2}}$. By assumption there is an $x$ such that $|AM| = 2x$ and $|MD| = x$.

Now the **power of a point theorem** (applied to the chords $AD$ and $BC$) implies that

$$2x \times x = |BM| \times |MC| = \frac{1}{2},$$

so that $x = \frac{1}{2}$ and $|AD| = \frac{3}{2}$. Now applying the Law of Cosines to the triangle $AOD$ we find

$$\left(\frac{3}{2}\right)^2 = 1^2 + 1^2 - 2(1)(1)\cos(\angle AOD)$$

which leads to the solution above.