Cal Poly Department of Mathematics

Puzzle of the Week

Jan 16-22, 2014

In the triangle ABC let M denote the midpoint of BC, let O denote the circumcenter, and let D denote the second point intersection of AM with the circumcircle of ABC.

Suppose we are given that $\angle BAC = 45^o$ and that $|AM| = 2|MD|$. Find $\cos(\angle AOD)$.

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u>

The cosine of $\angle AOD$ is $-\frac{1}{8}$.

Without loss of generality assume that the radius of the circumcircle is 1. An important observation is that since $\angle BAC = 45^o$ it follows that $\angle BOC = 90^o$. Therefore $|BC| = \sqrt{2}$ and $|BM| = |MC| = \frac{1}{\sqrt{2}}$. By assumption there is an x such that $|AM| = 2x$ and $|MD| = x$.

Now the *power of a point theorem* (applied to the chords AD and BC) implies that

$$2x \times x = |BM| \times |MC| = \frac{1}{2},$$

so that $x = \frac{1}{2}$ and $|AD| = \frac{3}{2}$. Now applying the Law of Cosines to the triangle AOD we find

$$\left(\frac{3}{2}\right)^2 = 1^2 + 1^2 - 2(1)(1)\cos(\angle AOD)$$

which leads to the solution above.