Cal Poly Department of Mathematics

Puzzle of the Week

Nov 14-20, 2013

Another past Putnam problem:

Let E denote the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and E. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and E.

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: This was on the 1994 Putnam exam as problem A2. The answer is $m = 2/9$.

For $a > 0$ let U_a denote the region bounded by $y = ax$, the y-axis, and E. Let L_α denote the region bounded by $y = ax$, the x-axis, and E. One can compute the areas of U_a and L_a directly, or one can simplify matters by using a change of variables: let ϕ denote the transformation $x = 3u,\ y = v$. Then $\phi^{-1}(E)$ is just the unit circle in the u, v plane, and if ℓ is the line $y = ax$ then $\phi^{-1}(\ell)$ is the line $v = 3au$. Now, ϕ is a linear transformation which magnifies all areas by a factor of $|\det \phi| = 3$. So

$$\text{area of } L_a = 3(\text{area of } \phi^{-1}(L_a)) = 3\frac{\arctan 3a}{2}$$

and

$$\text{area of } U_a = 3(\text{area of } \phi^{-1}(U_)) = 3\frac{\operatorname{arccot} 3a}{2}$$

So the area of $L_{1/2}$ is $\frac{3}{2}\arctan\frac{3}{2}$ while the area of U_m is $\frac{3}{2}\operatorname{arccot} 3m$. Equating the two and solving for m gives the answer above.