Cal Poly Department of Mathematics

Puzzle of the Week

Nov 7-13, 2013

In honor of the upcoming Putnam Exam competition here is an old competition problem to keep you busy over the long weekend:
A right triangle ABC has right angle at C and $\angle BAC = \theta$. The point D is chosen on AB so that $|AC| = |AD| = 1$. The point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0} |EF|$.

Solutions should be submitted to Morgan Sherman:

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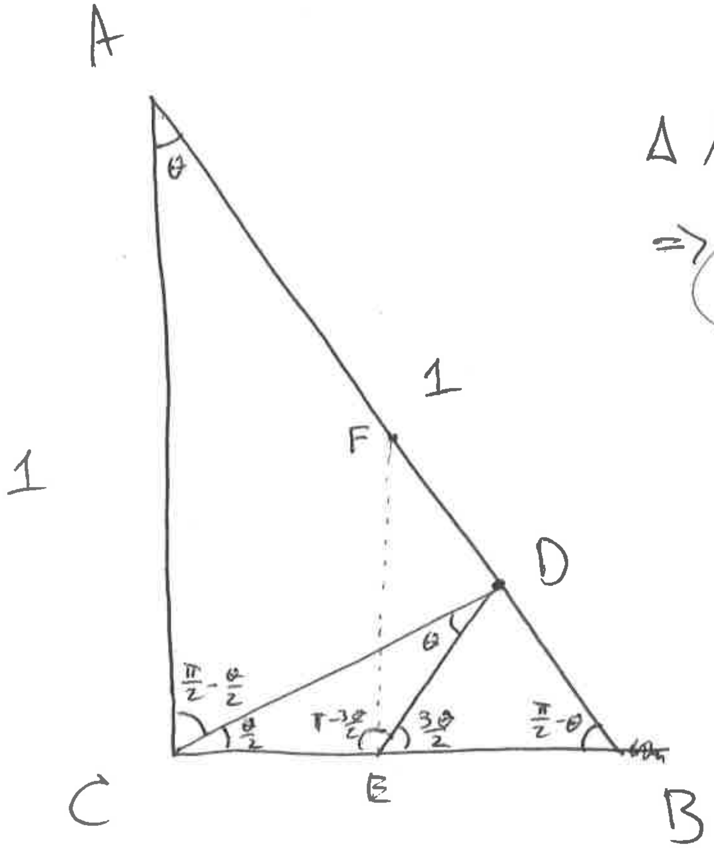
before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> This comes from the 1999 Putnam Exam, problem B1. The limits goes to $\frac{1}{3}$.

I have run out of time to type up my solution this week, but you can see my handwritten work on the next page:



$\Delta ACB \sim \Delta FEB$

$$\Rightarrow \frac{|EF|}{1} = \frac{|BE|}{|BC|}$$

Now: $|BC| = \tan\theta$

Law of Sines: $\dfrac{|CE|}{\sin\theta} = \dfrac{|CD|}{\sin\left(\pi - \frac{3\theta}{2}\right)}$ and $\dfrac{|CD|}{\sin\theta} = \dfrac{1}{\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$

$$\Rightarrow |CE| = \frac{\sin\theta}{\sin\frac{3\theta}{2}}|CD| = \frac{\sin\theta}{\sin\frac{3\theta}{2}} \cdot \frac{\sin\theta}{\cos\frac{\theta}{2}}$$

$$\therefore |EF| = \frac{|BE|}{|BC|} = \frac{|BC| - |CE|}{|BC|} = \frac{\tan\theta - \dfrac{\sin^2\theta}{\sin\frac{3\theta}{2}\cos\frac{\theta}{2}}}{\tan\theta}$$

$$= 1 - \frac{\sin\theta\cos\theta}{\sin\frac{3\theta}{2}\cos\frac{\theta}{2}} \longrightarrow 1 - \frac{2}{3} = \boxed{\frac{1}{3}} \text{ as } \theta \to 0$$