Cal Poly Department of Mathematics

Puzzle of the Week

October 3-9, 2013

Find a nonnegative integer n such that $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ is not an integer, or prove that there is no such n.

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> There is no such n.

There are a few ways to show this. Let $p(n) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

By induction: In fact after testing the first few values of n one might be led to the conjecture that $p(n) = 1^4 + 2^4 + \ldots + n^4 = \sum_{i=1}^n i^4$. Using induction and a lot of algebra establishes this. (Induction still works even if you didn't notice this alternate form for $p(n)$.

Alternatively some algebra leads one to the factorization:

$$p(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Now one shows the numerator is always divisible by 30.

Finally one can use modular arithmetic. $30p(n) = 6n^5 + 15n^2 + 10n^3 - n$ is certainly an integer and we work out that:

$$\begin{aligned} 30p(n) &\equiv n^2 - n \equiv n - n \equiv 0 \pmod 2 \\ 30p(n) &\equiv n^3 - n \equiv n - n \equiv 0 \pmod 3 \\ 30p(n) &\equiv n^5 - n \equiv n - n \equiv 0 \pmod 5 \end{aligned}$$

where in each line above, the second-to-last congruence is due to Fermat's Little Theorem. So $30p(n)$ is divisible by each of 2, 3, and 5, so it is divisible by 30. Hence $p(n)$ is an integer.