Cal Poly Department of Mathematics

Puzzle of the Week

May 2-15, 2013

From Tom O'Neil:

Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n^4}\prod_{i=1}^{2n}(n^2+i^2)^{1/n}\right).$$

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The limit evaluates to $25e^{2\arctan 2-4}$.

First rewrite the product within the limit as

$$a_n = \frac{1}{n^4}\prod_{i=1}^{2n}(n^2+i^2)^{1/n} = \prod_{i=1}^{2n}\left(\frac{n^2+i^2}{n^2}\right)^{1/n} = \prod_{i=1}^{2n}\left(1+\left(\frac{i}{n}\right)^2\right)^{1/n}.$$

So then

$$\log a_n = \sum_{i=1}^{2n}\frac{1}{n}\log\left(1+\left(\frac{i}{n}\right)^2\right),$$

and, recognizing this as a Riemann sum, we have

$$\lim_{n\to\infty}\log a_n = \int_0^2 \log(1+x^2)\,dx = \log 25 + 2\arctan 2 - 4.$$

Exponentiating gives us the result.