Cal Poly Department of Mathematics

Puzzle of the Week

April 18-24, 2013

Relayed from Bob Wolf:

Eight people sit at a circular table. They all stand up and sit in a new seat where the new seat is either their original seat, or adjacent to the original seat (on either side). In how many ways can they choose their new seats?

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

Solution: Note: This question was asked on NPR's "Sunday Puzzle" (March 3rd, this year) with Will Shortz (editor of the New York Times crossword puzzle). Most of these puzzles involve some sort of word-play, but this one was was a rare purely mathematical one.

In short there are 49 ways in which they can re-seat themselves. Here is one explanation on how to compute it.

Let C_n be the number of ways n people seated around a circular table can re-seat themselves according to the above rules. In addition let S_n be the number of ways n people seated in a straight line could re-seat themselves according to the same rules (here the two people on the ends have only two seats to choose from).

Now consider person A. If she sits back in her own seat then, viewing the remaining $n-1$ seats as if they were in a line, there are S_{n-1} ways for the others to sit.

Now suppose person A sits in the seat to her right where person B had been sitting. If person B also sits in the seat to his right, then all people will be forced to move to exactly the seat on their right. That is one possibility. Otherwise persons A and B will have swapped seats, and we see there will be S_{n-2} ways for the remaining people to choose their seats. Thus in total there are $S_{n-2}+1$ ways in which A can sit in the seat to her right. Similarly there are $S_{n-2}+1$ ways A can sit in the chair to her left.

So we have the formula $C_n = S_{n-1} + 2S_{n-2} + 2$. We now find a formula for S_k. For $k \geq 3$ consider a person at either end. If she sits in her own seat then there are S_{k-1} ways for the rest to sit. Otherwise she sits in her neighbor's seat and he must then sit in her seat (otherwise each person shifts down the line and the person at the other far end of the table has nowhere to sit). There are then S_{k-2} ways for the remaining people to sit in this case.

Therefore, for $k \geq 3$ we have $S_k = S_{k-1} + S_{k-2}$. It's easy to see that $S_1 = 1$ and $S_2 = 2$ (so in fact S_k is just the Fibonacci sequence, shifted by one index). Now we can quickly calculate:

k	1	2	3	4	5	6	7
S_k	1	2	3	5	8	13	21

Therefore $C_8 = S_7 + 2S_6 + 2 = 21 + 2 \cdot 13 + 2 = 49$.