Cal Poly Department of Mathematics

Puzzle of the Week

April 11-17, 2013

Relayed from Erin Pearse:

Let $a \geq 0$ be a fixed integer. Evaluate the limit

$$\lim_{n\to\infty} \sum_{k=0}^{an} \frac{k}{n^2+k^2}.$$

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: The limit converges to $\log\sqrt{a^2+1}$.

The trick is to view the limit as a Riemann sum:

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=0}^{an} \frac{k}{n^2+k^2} &= \lim_{n\to\infty} \sum_{k=0}^{an} \frac{k/n}{1+(k/n)^2}\,\frac{1}{n} \\ &= \int_0^a \frac{x}{1+x^2}\,dx \\ &= \frac{1}{2}\log(1+a^2)\end{aligned}$$

(I'll let the reader justify the details in the second equality...)