Cal Poly Department of Mathematics

Puzzle of the Week

Feb 21 - 27, 2013

Let a, b be real numbers in $(0, 1)$, each independently chosen at random. Compute the probability that the nearest integer to $\dfrac{a}{b}$ is odd. [Hint: the answer is *not* $\frac{1}{2}$.]

Solutions should be submitted to Morgan Sherman:

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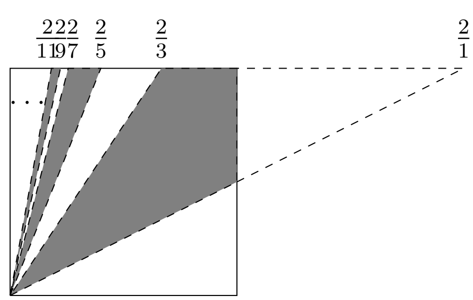
before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> The probability is $\frac{\pi-1}{4} = 0.535398\ldots$.

Let $(x, y) \in (0, 1) \times (0, 1)$. The ratio y/x is closer to an odd integer than an even integer if and only if there exists an integer n such that $(2n + \frac{1}{2})x < y < (2n + \frac{3}{2})x$. The set of such (x, y) is pictured below:



Now the probability we are looking for is the proportion of area in the shaded regions to the area of the square (which of course has area 1). From this picture, using a little geometry and algebra we calculate the area in the shaded regions to be

$$p = \left[\frac{1}{2}\left(\frac{2}{1} - \frac{2}{3}\right) - \frac{1}{4}\right] + \frac{1}{2}\left(\frac{2}{5} - \frac{2}{7}\right) + \frac{1}{2}\left(\frac{2}{9} - \frac{2}{11}\right) + \ldots = -\frac{1}{4} + \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

and the alternating sum is well-known to converge to $\pi/4$.