Cal Poly Department of Mathematics

Puzzle of the Week

Feb 14 - 20, 2013

From Tom O'Neil:

Compute

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \dots\right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots\right) \mathrm{d}\, x$$

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The integral converges to $\sqrt{e}$. Note that this problem appeared on the 1997 Putnam Exam.

The first series is just $xe^{-x^2/2}$. Using this we find the integral equals

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} \int_0^\infty x^{2n+1} e^{-x^2/2}\, dx$$

Now if $I_n = \int_0^\infty x^{2n+1} e^{-x^2/2}\, dx$ then integration by parts shows that $I_n = 2nI_{n-1}$. This together with $I_0 = 1$ shows that $I_n = 2^n n!$. Thus the above sum becomes

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} I_n = \sum_{n=0}^{\infty} \frac{2^n n!}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} = e^{1/2}.$$