Cal Poly Department of Mathematics

Puzzle of the Week

Jan 24 - 30, 2013

Find the minimum value of $f(x) = \frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$ for $0 < x < \frac{\pi}{2}$.

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: The minimum value is $f(\frac{\pi}{4}) = 1$.

First note that $f(x) = f(\frac{\pi}{2} - x)$ so it suffices to find the minimum on the interval $0 < x \leq \frac{\pi}{4}$. Now

$$f(x) = \frac{\sin^4 x + \cos^4 x}{\sin x \cos x} = \frac{\sin^2 x(1 - \cos^2 x) + \cos^2 x(1 - \sin^2 x)}{\sin x \cos x} = \frac{2}{\sin 2x} - \sin 2x.$$

Since $\sin 2x$ increases from 0 to 1 as x increases from 0 to $\pi/4$ we just need to maximize the function $g(u) = \frac{2}{u} - u$ for $0 < u \leq 1$. But $g'(u) = -\frac{2}{u^2} - 1 < 0$ so g strictly decreases and the minimum is $g(1) = f(\frac{\pi}{4}) = 1$