Cal Poly Department of Mathematics

Puzzle of the Week

Nov 8 - 14, 2012

Suppose $a_1, a_2, a_3, \ldots$ are positive numbers such that $\sum_{n=1}^{\infty} a_n = 5$. What are all possible values of $\displaystyle\sum_{n=1}^{\infty} a_n^2$?

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> This question appeared on the 2000 Putnam exam as problem A1. The series can sum to any value in the open interval $(0, 25)$.

First of all note that since all the terms a_n are positive we may calculate

$$25 = 5^2 = \left(\sum_{n=1}^{\infty} a_n\right)^2 = \sum_{n=1}^{\infty} a_n^2 + \sum_{k,l \geq 1} a_k a_l$$

and since the last summation is positive we find $\sum a_n^2 < 25$.

To see that any value in $(0, 25)$ can be attained we use geometric series. Let $N \in (0, 25)$. First of all suppose $a, r > 0$ satisfy $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = 5$. Then $\sum_{n=0}^{\infty} (ar^n)^2 = \frac{a^2}{1-r^2}$. So we need to be able to solve the two equations

$$\frac{a}{1-r} = 5 \text{ and } \frac{a^2}{1-r^2} = N.$$

After a little algebra these reduce to

$$r = \frac{25-N}{25+N} \text{ and } a = 25(1-r)$$

which has positive solutions r, a precisely when $0 < N < 25$.