Cal Poly Department of Mathematics

Puzzle of the Week

Nov 1 - 7, 2012

From Jason Lee:

Find, with proof, all positive integers which *cannot* be written as the sum of two or more consecutive positive integers. [For example $12 = 3 + 4 + 5$ but there is no such representation for 4.]

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> A number is not expressible as a sum of consecutive positive integers if and only if it is a power of two.

First note that any sum of $k > 1$ consecutive positive integers can be written as

$$n + (n+1) + (n+2) + \ldots + (n+k-1) = kn + \frac{k(k-1)}{2}.$$

If k is odd we can write this as $k(n + \frac{k-1}{2})$ which has an odd factor greater than one; if k is even we can write this as $\frac{k}{2}(2n + k - 1)$ which also has an odd factor greater than one.

Now conversely take a positive integer N with an odd factor, say $N = n(2k+1)$ where $n, k > 0$. If $n = 1$ then we can just write $N = 2k + 1 = (k) + (k+1)$. So suppose that $n > 1$ and note that

$$N = n(2k+1) = (n-k) + (n-k+1) + \ldots + (n) + \ldots + (n+k)$$

which writes N as a sum of consecutive integers. If some of these are negative (which will happen when $n < k$) then simply delete those, along with their canceling positive terms, to trim it to a sum of positive integers. Since $n > 1$ there will be at least two of them left.