Cal Poly Department of Mathematics

Puzzle of the Week

Relayed from Bob Wolf:

Wanda plays a game where in each turn she picks a uniform random real number \( x \in [0, 1] \). She continues taking turns until the sum of her numbers exceeds 1. If Wanda gets paid one dollar for every turn taken how much should she be willing to spend to play this game?

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle’s web site (see below) as well as in next week’s email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: [This problem (with different wording) appeared on the 1958 Putnam Exam.]

Wanda should be willing to spend \( e \) dollars to play such a game.

Note that the expected value of the game is \( \sum_{n=2}^{\infty} n \Pr(n) \) where \( \Pr(n) \) is the probability the game lasts exactly \( n \) rounds. Let \( T_n = \{(x_1, x_2, \ldots, x_n) \mid \sum x_i \leq 1\} \). Then note that \( \Pr(n) = \text{Vol}( \{(x_1, \ldots, x_{n-1}, x_n) \in [0, 1]^n \mid x_1 + \ldots + x_{n-1} \leq 1, \ x_n > 1 - (x_1 + \ldots + x_{n-1}) \}) \).

This can be calculated as

\[
\int_{T_{n-1}} \left[ \int_{1-(x_1+\ldots+x_{n-1})}^1 d(x_1, \ldots, x_{n-1}) \right] d(x_1, \ldots, x_{n-1}) = \sum_{i=1}^{n-1} \int_{T_{n-1}} x_i d(x_1, \ldots, x_{n-1}) = (n-1) \frac{1}{n!}
\]

(where I’ll leave the last equality as an exercise!). Now finally

\[
\sum_{n=2}^{\infty} n \Pr(n) = \sum_{n=2}^{\infty} \frac{n(n-1)}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = e
\]