Cal Poly Department of Mathematics

Puzzle of the Week

May 15-21, 2012

Consider the n-dimensional cube $C_n = [-2,2]^n$ (that is all points in $\mathbb{R}^n$ all of whose coordinates are between -2 and 2). At each of the points whose coordinates are each either ± 1 place a unit ball (there will be 2^n of them). Now let B_n denote the largest ball centered at the origin not intersecting any of these. Calculate

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(B_n)}{\mathrm{Vol}(C_n)}.$$

Note that you will need to look up the formula for the volume of a ball of radius R in n-dimensional space.

[By way of example, if $n = 2$ then we will place a unit disk at each of $(1,1), (1,-1), (-1,1), (-1,-1)$. We calculate that B_2 will have radius $\sqrt{2}-1$ and $\frac{\mathrm{Vol}(B_2)}{\mathrm{Vol}(C_2)} = \frac{\pi(\sqrt{2}-1)^2}{16} \approx 0.034$]

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Tuesday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: The limit diverges to ∞ (!)

I found this interesting phenomenon while browsing the research math web forum "math-overflow.net". I found it fascinating.

The formula for the volume of a n-dimensional ball of radius R is $\frac{\pi^{n/2}R^n}{\Gamma(1+\frac{n}{2})}$, where Γ denotes the famous "Gamma function". The radius of B_n is $\sqrt{n}-1$. Note that, counter-intuitively

B_n is not even contained in C_n once $\sqrt{n}-1>2$. Now

$$\frac{\mathrm{Vol}B_n}{\mathrm{Vol}C_n}=\frac{\pi^{n/2}(\sqrt{n}-1)^n}{4^n\Gamma(1+\frac{n}{2})}.$$

Using *Stirling's asymptotic formula*

$$\Gamma(z)=\sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1+O\left(\frac{1}{z}\right)\right)$$

and some algebra we find that

$$\lim_{n\to\infty}\frac{\mathrm{Vol}B_n}{\mathrm{Vol}C_n}=\lim_{n\to\infty}\left(\frac{\sqrt{2\pi e}}{4}\cdot\frac{\sqrt{n}-1}{\sqrt{n+2}}\right)^n\cdot\frac{1}{\sqrt{n+2}}\cdot\frac{e}{\sqrt{\pi}}$$

Now one checks that $\sqrt{2\pi e}=4.132731...>4$, therefore for large n the term inside the parentheses will be strictly larger than (and can be bounded away from) one, meaning it will grow exponentially and beat out the $\frac{1}{\sqrt{n+2}}$ outside.

One has to go to a very large dimensional space before the ratio of volumes is even greater than 1. I calculate (using the computer algebra system *sage*) that at about dimension 1206 is the first time that B_{1206} has larger volume than C_{1206}.