Cal Poly Department of Mathematics

Puzzle of the Week

May 1-8, 2012

From Kent Morrison:

Let n be a positive integer. For each integer $1 \leq k \leq n$ we place k balls labeled "k" into a bag. Now for each such k choose a ball at random and let $f(k)$ denote its label (each ball is replaced before the next ball is chosen). Let d_n denote the probability that $f(k) \neq k$ for each k. Compute

$$\lim_{n\to\infty} d_n.$$

Solutions should be submitted to Morgan Sherman:

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before next Tuesday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The limit converges to $\frac{1}{e}$.

I received a few different and very nice approaches. Here is one heuristic argument I made up, which I will let the reader make rigorous. Let $T_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. Then it is not hard to see that

$$d_n = \frac{(T_n - 1)(T_n - 2)\ldots(T_n - n)}{T_n^n} = \left(1 - \frac{1}{T_n}\right)\left(1 - \frac{2}{T_n}\right)\ldots\left(1 - \frac{n}{T_n}\right)$$

Using $\log(1 - x) = -(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots)$ for $|x| < 1$ we find

$$\begin{aligned}\log d_n &= -\sum_{k=1}^{n}\left(\frac{k}{T_n} + \frac{1}{2}\frac{k^2}{T_n^2} + \frac{1}{3}\frac{k^3}{T_n^3} + \ldots\right)\\ &= -\left(\frac{1 + 2 + \ldots + n}{T_n} + \frac{1}{2}\frac{1^2 + 2^2 + \ldots + n^2}{T_n^2} + \frac{1}{3}\frac{1^3 + 2^3 + \ldots + n^3}{T_n^3} + \ldots\right)\end{aligned}$$

Now argue that $\lim_{n\to\infty} \log d_n = -1$ to finish the proof.