Cal Poly Department of Mathematics

Puzzle of the Week

April 17-23, 2012

From Kent Morrison:

Let D denote a rectangle (with width a and height b, say) which is centered on the origin and has sides parallel to the axes. Suppose that n points are chosen (uniformly) randomly inside D. What is the probability P_n that they all lie on the same side of some line through the origin?

Solutions should be submitted to Morgan Sherman:

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before next Tuesday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The probability is $\frac{n}{2^{n-1}}$.

A more common variation of this problem has all the points inside a circle and asks what is the probability that all n points lie in some semi-circle. As it turns out the only important feature of the circle is that any line through its center divides it into two equal areas. The same is true for the rectangle in our problem so the same result holds. What follows is my paraphrased version of Kent Morrison's solution, which uses some standard probability notation:

Denote the points $P_1, P_2, \ldots, P_n$ and let E be the event that all points lie on one side of a line through the origin. Let ℓ_i denote the line through the origin and P_i; if an observer stands at the origin and faces P_i then let S_i be the half of the rectangle left of and including ℓ_i as viewed by the observer.

Now let E_i be the event that all points lie in S_i. Note that $P(E_i) = \frac{1}{2^{n-1}}$. One checks that E_i and E_j are disjoint for $i \neq j$ and furthermore that $E = \cup E_i$. Therefore

$$P(E) = \sum P(E_i) = \frac{n}{2^{n-1}}.$$