Cal Poly Department of Mathematics

Puzzle of the Week

April 3-9, 2012

Let f_n denote the nth Fibonacci number, defined by $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Evaluate the sum

$$\sum_{n=0}^{\infty} \frac{f_n}{2012^n}$$

Solutions should be submitted to Morgan Sherman:

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before next Tuesday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: The series sums to $\frac{2012}{2012^2-2012-1} = \frac{2012}{4046131}$.

First of all the series converges since, for example, $f_n < 2^n$ by induction and we can compare to a convergent geometric series. Let $s = \frac{f_0}{1} + \frac{f_1}{2012} + \frac{f_2}{2012^2} + \ldots$ denote the sum. Then

$$\begin{aligned} s + 2012s &= 2012f_0 + \frac{f_0 + f_1}{1} + \frac{f_1 + f_2}{2012} + \frac{f_2 + f_3}{2012^2} + \ldots \\ &= 0 + 2012^2 s - 2012 f_1. \end{aligned}$$

Solving for s we find the solution above.

Note: we can also take the known closed form expression $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ and sum two geometric series to get the same answer.