Cal Poly Department of Mathematics

Puzzle of the Week

Feb 16-22, 2012

Fix a positive number $a > 0$. What is the supremum (i.e. *least upper bound*) of

$$\frac{1}{a^x} + \frac{1}{a^{1/x}} \qquad \text{for } x > 0 \text{ ?}$$

Solutions should be submitted to Morgan Sherman:

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before next Thursday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The supremum is: ∞ if $a < 1$; and $\max\{\frac{2}{a}, 1\}$ if $a \geq 1$.

Let $f(x) = a^{-x} + a^{-\frac{1}{x}}$. First note that if $0 < a < 1$ then $f(x) \to \infty$ as $x \to 0^+$.

Assume $a > 1$ and set $f(0) = 1$. Since $f(x) = f(1/x)$ we may take $0 \leq x \leq 1$ on which f is continuous. Therefore its maximum occurs either at an endpoint $f(0) = 1$, $f(1) = \frac{2}{a}$, or at a point $0 < x < 1$ where $f'(x) = 0$. The latter happens iff

$$f'(x) = \log a \left(-a^{-x} + \frac{1}{x^2} a^{-\frac{1}{x}} \right) = 0$$

or equivalently, at an x such that $a^{-\frac{1}{x}} = x^2 a^{-x}$. But at such an x we have $f(x) = \frac{1+x^2}{a^x}$. Let $g(x) = \frac{1+x^2}{a^x}$. Then

$$g''(x) = \frac{(\log a)^2}{a^x} \left[\left(x - \frac{2}{\log a} \right)^2 + 1 - \frac{2}{(\log a)^2} \right]$$

which is positive for $a > 1$ and $0 < x < 1$ (this is clear if $\log a > \sqrt{2}$; if $0 < \log a < \sqrt{2}$ then one shows the term in the brackets can only be negative for $x > 1$). Therefore $g(x)$ is concave up and takes its maximum value at an endpoint $x = 0, 1$. Since $f(x)$ takes the same value as $g(x)$ at these endpoints the solution follows.