For an $n \times n$ matrix $A$ define $\sin A$ by the convergent power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$$

For real numbers $x, y$ evaluate explicitly

$$\sin \left( \begin{array}{cc} x & y \\ 0 & x \end{array} \right)$$

Solutions should be submitted to Morgan Sherman:

\begin{quote}
Dept. of Mathematics, Cal Poly  
Email: sherman1 -AT- calpoly.edu  
Office: bldg 25 room 310
\end{quote}

before next Friday. Those with correct and complete solutions will have their names listed on the puzzle’s web site (see below) as well as in next week’s email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

\begin{quote}
Solution: We calculate that $\sin A = \left( \begin{array}{cc} \sin x & y \cos x \\ 0 & \sin x \end{array} \right)$. 
\end{quote}
To see this we calculate

\[
\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix}^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \begin{pmatrix} 1 & (2n+1) \frac{y}{x} \\ 0 & 1 \end{pmatrix}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} x^{2n+1} & (2n+1)x^{2n}y \\ 0 & x^{2n+1} \end{pmatrix}
\]

\[
= \begin{pmatrix} \sin x & y \cos x \\ 0 & \sin x \end{pmatrix}
\]

I found this problem as part of the solution to problem B4 of the 1996 Putnam Exam. The full problem asks whether or not there exists a 2 × 2 matrix \( A \) such that \( \sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix} \). One can reason as follows: if \( A \) is diagonalizable then so is \( \sin A \), so if \( A \) has distinct eigenvectors then \( \sin A \) cannot have the desired form. Otherwise \( A \) must have a repeated eigenvector and thus can be conjugated into the form given in the Puzzle of the Week problem. Then the calculation above shows that \( \sin A \) cannot have the desired form since \( \sin x = 1 \) implies that \( \cos x = 0 \).