Cal Poly Department of Mathematics

Puzzle of the Week

November 10-16, 2011

For an $n \times n$ matrix A define $\sin A$ by the convergent power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$$

For real numbers x, y evaluate explicitly

$$\sin \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: We calculate that $\sin A = \begin{pmatrix} \sin x & y\cos x \\ 0 & \sin x \end{pmatrix}$.

To see this we calculate

$$
\begin{aligned}
\sin A &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix}^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \begin{pmatrix} 1 & (2n+1)\frac{y}{x} \\ 0 & 1 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \begin{pmatrix} x^{2n+1} & (2n+1)x^{2n}y \\ 0 & x^{2n+1} \end{pmatrix} \\
&= \begin{pmatrix} \sin x & y\cos x \\ 0 & \sin x \end{pmatrix}
\end{aligned}
$$

I found this problem as part of the solution to problem B4 of the 1996 Putnam Exam. The full problem asks whether or not there exists a 2×2 matrix A such that $\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}$. One can reason as follows: if A is diagonalizable then so is $\sin A$, so if A has distinct eigenvectors then $\sin A$ cannot have the desired form. Otherwise A must have a repeated eigenvector and thus can be conjugated into the form given in the Puzzle of the Week problem. Then the calculation above shows that $\sin A$ cannot have the desired form since $\sin x = 1$ implies that $\cos x = 0$.