Cal Poly Department of Mathematics

Puzzle of the Week

November 3-9, 2011

Evaluate (and justify):

$$\int_0^\pi \log(\sin x)\,dx.$$

Here $\log t$ represents the natural logarithm of t.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The integral evaluates to $-\pi\log 2$.

Since $\sin(\pi - x) = \sin x$ and $\sin(\frac{\pi}{2} - x) = \cos x$ we can see that $\int_0^\pi \log(\sin x)\,dx = \int_0^\pi \log(\sin\frac{x}{2})\,dx = \int_0^\pi \log(\cos\frac{x}{2})\,dx$. Now let I be the integral in question. Then

$$\begin{aligned} I = \int_0^\pi \log(\sin x)\,dx &= \int_0^\pi \log\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right) \\ &= \int_0^\pi \log 2\,dx + \int_0^\pi \log\sin\frac{x}{2}\,dx + \int_0^\pi \log\cos\frac{x}{2}\,dx \\ &= \pi\log 2 + I + I \end{aligned}$$

Therefore $I = -\pi\log 2$.

Kent Morrison points out the following to me: Using the above we can find $\int_0^{\pi/2} \log(\sin^2 x)\,dx$ and therefore also $\int_0^{\pi/2} \log(1-\cos^2 x)\,dx$. Expanding the integrand as a series and using the reduction formulas for integrals of powers of cosines one can find

$$\log 2 = \frac{1}{4}\left(1 + \frac{1}{2}\frac{3}{4} + \frac{1}{3}\frac{3\cdot 5}{4\cdot 6} + \ldots + \frac{1}{n}\frac{3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot (2n)} + \ldots\right),$$

This is attributed to Nielsen in the 1890s (www.plouffe.fr/simon/articles/log2.pdf).