Cal Poly Department of Mathematics

Puzzle of the Week

October 20 - 26, 2011

Find, for any positive $\alpha \neq 1$, the exact value of the sum

$$\sum_{n=0}^{\infty} \frac{1}{\alpha^{2^n} - \alpha^{-2^n}}$$

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The summation converges to $1/(\alpha - 1)$ if $\alpha > 1$; it converges to $\alpha/(\alpha - 1)$ if $0 < \alpha < 1$.

My solution was to show by induction on N that

$$\sum_{n=0}^{N} \frac{1}{\alpha^{2^n} - \alpha^{-2^n}} = \frac{1}{\alpha - 1} \frac{\alpha^{2^{N+1}} - \alpha}{\alpha^{2^{N+1}} - 1}$$

from which the result above follows by letting $N \to \infty$. However some of the submitted solutions found the following nice argument. For $\alpha > 1$ we find that

$$\sum_{n=0}^{\infty} \frac{1}{\alpha^{2^n} - \alpha^{-2^n}} = \sum_{n=0}^{\infty} \frac{1}{\alpha^{2^n}} \frac{1}{1 - \alpha^{-2^{n+1}}} = \sum_{n=0}^{\infty} \frac{1}{\alpha^{2^n}} \sum_{k=0}^{\infty} \alpha^{-2^{n+1}k}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{\alpha}\right)^{2^n(2k+1)} = \sum_{l=1}^{\infty} \left(\frac{1}{\alpha}\right)^{l} = \frac{\frac{1}{\alpha}}{1 - \frac{1}{\alpha}}$$

since the exponenet $2^n(2k+1)$ runs over all positive integers as $n, k = 0, 1, 2, \ldots, \infty$. (Note that all the sums are absolutely convergent.) For the case $0 < \alpha < 1$ use the above formula with $1/\alpha$ in place of α.