Cal Poly Department of Mathematics

Puzzle of the Week

October 6 - October 12, 2011

Relayed to me from Vince Bonini:

Let f be a continuous function on $[0,1]$. Calculate (in terms of f)

$$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx.$$

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The limit converges to $f(0)\frac{\pi}{2}$.

Thanks to Vince Bonini for giving me the following very nice argument: Note first that $\arctan(nx)$ is an antiderivative of $\frac{n}{1+n^2x^2}$. Also recall the Mean Value Theorem for integrals: If $g(x)$ does not change sign on $[c,d]$ and both f,g are continuous on $[c,d]$ then there is an $a \in (c,d)$ such that $\int_c^d f(x)g(x)\,dx = f(a)\int_c^d g(x)\,dx$. Therefore for any $n > 1$ there are a_n, b_n satisfying

$$0 < a_n < \frac{1}{\sqrt{n}} < b_n < 1$$

such that

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx + \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx$$

$$= f(a_n)\arctan(\sqrt{n}) \; + \; f(b_n)\left(\arctan(n) - \arctan(\sqrt{n})\right).$$

Since f is continuous on $[0,1]$ it is bounded as well, so the second term goes to 0 as n goes to infinity, while the first approaches the solution above.