

Cal Poly Department of Mathematics

Puzzle of the Week

September 29 - October 5, 2011

Does the series $\sum_{n=0}^{\infty} \sin(\pi\sqrt{n^2+1})$ converge? Justify your solution.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

<http://www.calpoly.edu/~sherman1/puzzleoftheweek>

Solution: The series converges.

I borrowed this problem from an old University of Illinois at Urbana Champagne undergraduate math contest. Let $\epsilon_n = \sqrt{n^2+1} - n$. Since

$$\epsilon_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1} + n}$$

it is clear that the sequence $\{\epsilon_n\}$ decrease to a limit of 0. Therefore the same is true of $\{\sin(\pi\epsilon_n)\}$ once $\epsilon_n < 1/2$ (since $\sin(x)$ is increasing on $(0, \pi/2)$). Now

$$\sin(\pi\sqrt{n^2+1}) = \sin(\pi(n + \epsilon_n)) = \cos(\pi n) \sin(\pi\epsilon_n) = (-1)^n \sin(\pi\epsilon_n)$$

so the Alternating Series Test applies and the series is convergent.