Cal Poly Department of Mathematics

Puzzle of the Week

Apr 18-29, 2011

An imaginary line ℓ runs east-west and divides a large field in two. Land north of ℓ is worth 10 dollars per square foot while land south of ℓ is worth only 4 dollars per square foot. There is an oak tree growing 20ft south of ℓ and enclosed by 100ft of fence. To the nearest whole dollar what is the maximum possible value of the land within the fence?

Notes: Take the oak tree to be a single point; you might as well assume the fence begins and ends at the tree. In your solution proof is not required for credit.

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

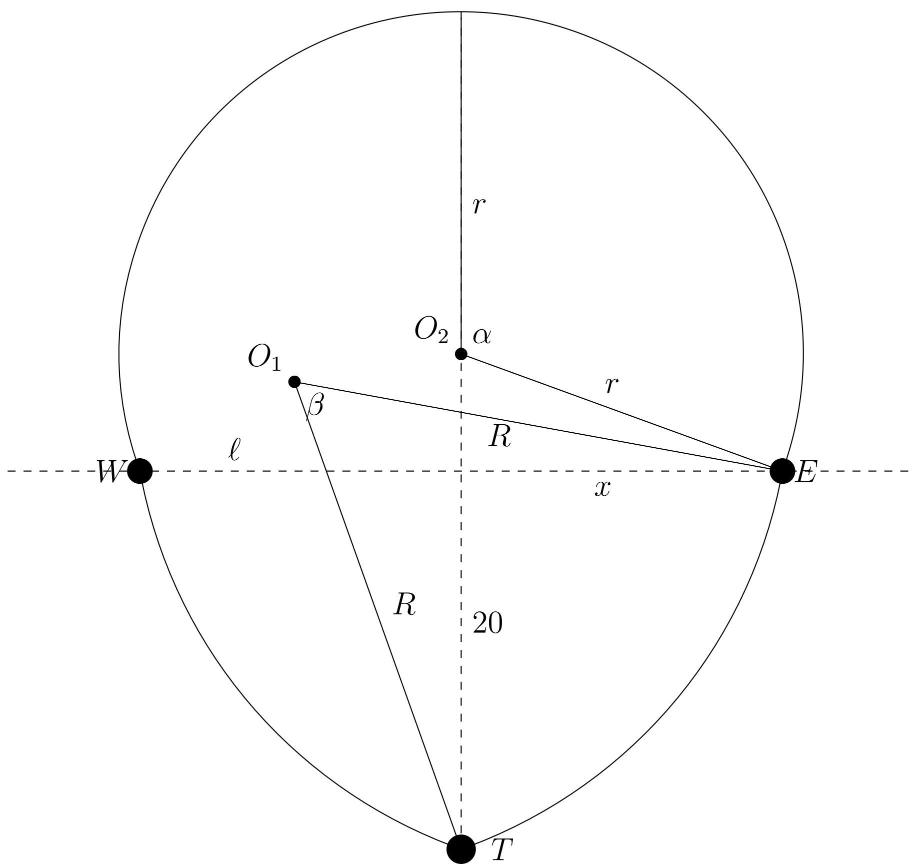
http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: The maximum value is approximately \$5357.

I took this problem from an old "IBM Ponder This" challenge. There were several different attempts at maximizing this land value from various solvers. Some placed a tiny amount of fence around the tree, then ran the two lines of fence up to the x-axis and made a circle with the remaining 60ft of fence. But this gives a land value of less than \$3000. Others enclosed everything in one big circle whose southern most point hits the tree. This gives a land value just shy of \$4800. One student cleverly matched two semi-ellipses, one south of ℓ, another north of ℓ to break the \$5000 barrier. A popular approach was to connect the tree to the line ℓ with straight line fences, then connect the fence with a semi-circle in the land north of ℓ. This "ice cream cone" shape manages to capture about \$5105.

In fact once can still do better. For example in the ice cream cone diagram if one uses not a semi-circle but a larger portion of a circle, just big enough to get the tangent lines to match at ℓ, one can get \$5128. In fact one can find the optimal ice cream cone shape and can manage a total land value around \$5220. However we can even do better than this.

Let T denote the tree and W, E denote the points that the fence crosses ℓ, on the west and east sides respectively. We use the following principle: *In areas of constant land value any free fencing should form an arc of a circle* (which always contains maximal area for given perimeter). Thus the fencing from T to E will be a circular arc, say from the circle of radius R centered at O_1, with subtended angle β. Similarly the fencing connecting E to W in the land north of ℓ will form an arc of a circle, say of radius r and subtended angle 2α and centered at O_2. The fence connecting W to T is, by symmetry, the mirror image of that connecting T to E.



From the diagram we get the equations

$$400 + x^2 = 4R^2 \sin^2\left(\frac{\beta}{2}\right), \quad x = r\sin(\alpha) \tag{1}$$

from which we can solve for R and r:

$$R = \frac{\sqrt{400 + x^2}}{2\sin(\beta/2)}, \quad r = \frac{x}{\sin(\alpha)} \tag{2}$$

Now we also have the constraint of 100ft of fencing, which means that

$$2\beta R + 2\alpha r = 100. \tag{3}$$

Substituting equations (2) into equation (3) yields a quadratic in x which can be solved explicitly (I'd suggest using a computer algebra system... it is not pretty). So all of the variables can be expressed in terms of the angles α and β.

Now the value of the land is

$$\begin{aligned} V = V(\alpha, \beta) = 2\left(\frac{b}{2}R^2 - \frac{1}{2}\sqrt{400 + x^2}R\cos\left(\frac{\beta}{2}\right) + 10x\right) \times 4 \\ + \big(\alpha r^2 + xr\cos(\pi - \alpha)\big) \times 10. \end{aligned}$$

This function of two variables can be maximized on the domain $[0, \pi] \times [0, \pi]$. I used *sage* to do this numerically and found

$$\alpha \approx 1.81 \approx 104^o, \quad \beta \approx 0.71 \approx 40^o$$

with a land value, to the nearest dollar, of \$5357. One can also check that with these values for α and β the circle arcs have equal tangents at the line ℓ, as one might expect from physical intuition.

This problem (as Mike Robertson points out) is really a physical "soap bubble" problem: we want to find a configuration for our one-dimensional film containing air with two different air pressures (or 'land value' in our case).