Cal Poly Department of Mathematics

Puzzle of the Week

Apr 1-7, 2011

For a real number x let $\{x\}$ denote the nearest integer to x, with the convention that we round down in case x is half-way between two consecutive integers. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{\{\sqrt{n}\}^3}.$$

You may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> The series sums to $\pi^2/3$.

Since $\{\sqrt{n}\} = a$ if and only if $a - \frac{1}{2} < \sqrt{n} \le a + \frac{1}{2}$, which is equivalent to $a^2 - a + \frac{1}{4} < n \le a^2 + a + \frac{1}{4}$. Therefore $\{\sqrt{n}\} = a$ iff $n = a^2 - a + 1, a^2 - a + 2, \ldots, a^2 + a$. There are $2a$ total of these so that

$$\sum_{n=1}^{\infty} \frac{1}{\{\sqrt{n}\}^3} = \sum_{a=1}^{\infty} 2a\frac{1}{a^3} = 2\frac{\pi^2}{6}.$$