Cal Poly Department of Mathematics

Puzzle of the Week

Feb 18-24, 2011

From Tom O'Neil:

Let $p(x) = 2 + 4x + 3x^2 + 5x^3 + 3x^4 + 4x^5 + 2x^6$ and for $y \in (0, 5)$ define

$$I(y) = \int_0^\infty \frac{x^y}{p(x)} dx.$$

For which y is $I(y)$ a minimum?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> The minimum occurs at $y = 2$.

As usual to find the minimum we compute compute the derivative and set equal to zero. Here differentiation under the integral sign is allowed and we find:

$$I'(y) = \int_0^\infty \frac{x^y \log x}{p(x)} dx = \int_0^1 \frac{x^y \log x}{p(x)} dx + \int_1^\infty \frac{x^y \log x}{p(x)} dx = \int_0^1 \frac{x^y \log x}{p(x)} dx - \int_0^1 \frac{u^{4-y} \log u}{p(u)} du$$
$$= \int_0^1 \frac{(x^y - x^{4-y}) \log x}{p(x)} dx$$

where we have made the substitution $u = 1/x$ and used the symmetry of the polynomial p, namely that $u^6 p(1/u) = p(u)$. The above integral is zero if and only if $y = 2$. Finally note that $I''(y) = \int_0^\infty \frac{x^y (\log x)^2}{p(x)} dx$ is strictly positive on the interval $0 < y < 5$, so by the second derivative test $y = 2$ is an absolute minimum.