Cal Poly Department of Mathematics

Puzzle of the Week

Jan 28-Feb 4, 2011

Calculate

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{i^2 j}{5^i(j5^i+i5^j)}$$

[Hint: Write $S+S$ is a clever way, then find and make use of a formula for $\sum_{i=1}^{\infty}\frac{i}{5^i}$]

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The series sums to 25/512.

This problem was relayed to me by Jonathan Shapiro who remembered it from the '99 Putnam exam (problem A4). Note that all terms are positive so we may rearrange terms. Setting S to be the value of the sum we calculate

$$2S = S+S = \sum_{i,j}\frac{i^2 j}{5^i(j5^i+i5^j)} + \sum_{i,j}\frac{ij^2}{5^j(i5^j+j5^i)} = \sum_{i,j}\frac{ij}{j5^i+i5^j}\left(\frac{i}{5^i}+\frac{j}{5^j}\right) =$$

$$= \sum_{i,j}\frac{ij}{j5^i+i5^j}\frac{j5^i+i5^j}{5^{i+j}} = \sum_{i,j}\frac{ij}{5^{i+j}} = \sum_i \frac{i}{5^i}\sum_j\frac{j}{5^j}$$

Now setting $f(x)=\sum_i x^i = 1/(1-x)$ for $|x|<1$ we find that

$$\frac{x}{(1-x)^2} = xf'(x) = x\sum_i ix^{i-1} = \sum_i ix^i, \quad |x|<1.$$

and therefore

$$2S = \frac{1/5}{(1-1/5)^2}\frac{1/5}{(1-1/5)^2} = \frac{25}{256}$$

which leads to the solution above.