Calculate
\[ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{i^2 j}{5^i (j 5^i + i 5^j)} \]

[Hint: Write \( S + S \) is a clever way, then find and make use of a formula for \( \sum_{i=1}^{\infty} \frac{i}{5^i} \)]

**Solution:** The series sums to \( \frac{25}{512} \).

This problem was relayed to me by Jonathan Shapiro who remembered it from the ’99 Putnam exam (problem A4). Note that all terms are positive so we may rearrange terms. Setting \( S \) to be the value of the sum we calculate

\[
2S = S + S = \sum_{i,j} \frac{i^2 j}{5^i (j 5^i + i 5^j)} + \sum_{i,j} \frac{ij^2}{5^i (i 5^i + j 5^j)} = \sum_{i,j} \frac{ij}{j 5^i + i 5^j} \left( \frac{i}{5^i} + \frac{j}{5^j} \right) = \\
= \sum_{i,j} \frac{ij}{j 5^i + i 5^j} \frac{j 5^i + i 5^j}{5^i+j} = \sum_{i,j} \frac{ij}{5^i+j} = \sum_{i} \frac{i}{5^i} \sum_{j} \frac{j}{5^j}
\]

Now setting \( f(x) = \sum_{i} x^i = 1/(1 - x) \) for \( |x| < 1 \) we find that

\[
\frac{x}{(1-x)^2} = xf'(x) = x \sum_{i} ix^{i-1} = \sum_{i} ix^{i}, \quad |x| < 1.
\]

and therefore

\[
2S = \frac{1/5}{(1 - 1/5)^2} \frac{1/5}{(1 - 1/5)^2} = \frac{25}{256}
\]

which leads to the solution above.