Cal Poly Department of Mathematics

Puzzle of the Week

Jan 21-27, 2011

From Tom O'Neil:

Recall that the *mean* $\overline{x}$ and *standard deviation* σ of a set of numbers $\{x_1, x_2, \ldots, x_k\}$ are defined by

$$\overline{x} = \frac{\sum_{i=1}^k x_i}{k}, \quad \sigma = \sqrt{\frac{\sum_{i=1}^k (x - \overline{x})^2}{k}}.$$

As it happens, for every set of seven consecutive integers the numbers $\overline{x}$ and σ are also integers. [For example the mean of $\{6, 7, 8, 9, 10, 11, 12\}$ is 9 while it's standard deviation is 2]

What is the next value of k for which any set of k consecutive integers have both $\overline{x}$ and σ integers?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/˜sherman1/puzzleoftheweek

<u>*Solution:*</u> The next k which works is $k = 97$.

First note that for $\overline{x}$ to be an integer we must have k odd. Setting $k = 2l + 1$ we have that any set of k consecutive integers are $\overline{x} - l, \overline{x} - l + 1, \ldots, \overline{x}, \ldots, \overline{x} + l$. Then we calculate

$$\sigma = \sqrt{\frac{2\sum_{j=0}^l j^2}{k}} = \sqrt{\frac{l(l+1)}{3}}.$$

Setting what is under the radical equal to a perfect square n^2 and solving for l we find that

$$l = \frac{\sqrt{12n^2 + 1} - 1}{2}.$$

Therefore we have a an integer value for σ precisely when $12n^2 + 1 = m^2$, which, as Tom O'Neil points out is an example of Pell's equation. One can look for solutions by hand, or solve it generally, to find the answer for k above.