Cal Poly Department of Mathematics

Puzzle of the Week

Nov 12 - 18, 2010

Let $f(x) = e^{-x^2}$. Find a function $g \not\equiv 0$ and an interval (a, b) as large as possible containing zero, on which $g(x)$ is defined, such that the false product rule

$$(fg)' = f'g'$$

holds.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

Solution: We can take $g(x) = K\dfrac{e^x}{\sqrt{2x+1}}$ (where K is any nonzero constant) on the interval $(-\frac{1}{2}, \infty)$.

This is the last in the "Putnam warm-up series". This problem comes (almost identically) from the 1988 Putnam Exam (problem A2). The differential equation $(fg)' = f'g'$ can be rewritten as

$$g' + \frac{-2x}{2x+1}g = 0.$$

By the existence and uniqueness theorem for first order ODEs there is a solution defined on any interval not containing $-\frac{1}{2}$. Separating variables one finds

$$\frac{g'}{g} = \frac{2x}{2x+1} = 1 - \frac{1}{2x+1}$$

Now integrating both sides gives $\log|g(x)| = x - \frac{1}{2}\log|2x+1| + C$, and after exponentiating we arrive at the solution above.