Cal Poly Department of Mathematics

Puzzle of the Week

Oct 29 - Nov 4, 2010

Calculate, with justification, the integral

$$\int_2^3 \frac{\sin\sqrt{13-x}}{\sin\sqrt{13-x}+\sin\sqrt{8+x}}\,dx$$

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

http://www.calpoly.edu/~sherman1/puzzleoftheweek

<u>*Solution:*</u> The value of the integral is 1/2.

This problem is taken from (with modifications) problem B1 of the 1987 Putnam Exam. The trick here is to use some symmetry: the change of variables $u = 5 - x$ transforms $13 - x$ into $8 + u$ and vice-versa. At the same time as x ranges over $[2, 3]$ so does u. Hence we try the substitution:

$$\begin{aligned} I &= \int_2^3 \frac{\sin\sqrt{13-x}}{\sin\sqrt{13-x}+\sin\sqrt{8+x}}\,dx \\ &= -\int_3^2 \frac{\sin\sqrt{8+u}}{\sin\sqrt{13-u}+\sin\sqrt{8+u}}\,du \\ &= \int_2^3 \frac{\sin\sqrt{8+x}}{\sin\sqrt{13-x}+\sin\sqrt{8+x}}\,dx \end{aligned}$$

and therefore

$$I + I = \int_2^3 \frac{\sin\sqrt{13-x}+\sin\sqrt{8+x}}{\sin\sqrt{13-x}+\sin\sqrt{8+x}}\,dx = \int_2^3 dx = 1.$$

Hence the answer above.