

Cal Poly Department of Mathematics

Puzzle of the Week

Oct 15 - 21, 2010

From Jeremy Kun:

Suppose x satisfies $x + \frac{1}{x} = -1$. What is the value of $x^{65} + \frac{1}{x^{65}}$?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed on the puzzle's web site (see below) as well as in next week's email announcement. Anybody is welcome to make a submission.

<http://www.calpoly.edu/~sherman1/puzzleoftheweek>

Solution: The answer is -1 .

Here is one approach: Clearing the fraction we find that $x^2 + x + 1 = 0$. Multiplying both sides by $(x - 1)$ we get $x^3 = 1$. Therefore $x^{65} = (x^3)^{21}x^2 = x^2$ and

$$x^{65} + \frac{1}{x^{65}} = x^2 + \frac{1}{x^2}.$$

But by squaring the original equation we find

$$\left(x + \frac{1}{x}\right)^2 = (-1)^2 \implies x^2 + 2 + \frac{1}{x^2} = 1 \implies x^2 + \frac{1}{x^2} = -1$$

hence the answer above.

Equivalently we can solve $x^2 + x + 1 = 0$ to find that $x = e^{\pm i\frac{\pi}{3}}$, i.e. x is a cube root of unity different from 1. Now one can calculate $x^{65} + \frac{1}{x^{65}}$ directly.