Cal Poly Department of Mathematics

Puzzle of the Week

May 21-27, 2010

Calculate the integral:

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^{\sqrt{5}}}$$

[Hint: $\tan(\pi/2 - u) = \cot(u)$]

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution:

The value of the integral is $\frac{\pi}{4}$.

This problem appeared on the 1980 Putnam Exam (without the hint, and with an exponent of $\sqrt{2}$ instead of $\sqrt{5}$). Following the hint we make the substitution $u = \frac{\pi}{2} - x$ to find

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^{\sqrt{5}}} = -\int_{\frac{\pi}{2}}^0 \frac{du}{1+(\cot u)^{\sqrt{5}}} = \int_0^{\frac{\pi}{2}} \frac{(\tan u)^{\sqrt{5}}\,du}{1+(\tan u)^{\sqrt{5}}}$$

and therefore

$$2I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^{\sqrt{5}}} + \int_0^{\frac{\pi}{2}} \frac{(\tan x)^{\sqrt{5}}\,dx}{1+(\tan x)^{\sqrt{5}}} = \int_0^{\frac{\pi}{2}} \frac{1+(\tan x)^{\sqrt{5}}}{1+(\tan x)^{\sqrt{5}}}\,dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$

Hence the answer above.