Puzzle of the Week
May 21-27, 2010

Calculate the integral:

\[ \int_0^{\pi/2} \frac{dx}{1 + (\tan(x))^{\sqrt{5}}} \]

[Hint: \( \tan(\pi/2 - u) = \cot(u) \)]

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week’s email announcement. Anybody is welcome to make a submission.

Solution:
The value of the integral is \( \frac{\pi}{4} \).

This problem appeared on the 1980 Putnam Exam (without the hint, and with an exponent of \( \sqrt{2} \) instead of \( \sqrt{5} \)). Following the hint we make the substitution \( u = \frac{\pi}{2} - x \) to find

\[ I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{5}}} = - \int_0^{\pi/2} \frac{du}{1 + (\cot u)^{\sqrt{5}}} = \int_0^{\pi/2} \frac{(\tan u)^{\sqrt{5}}}{1 + (\tan u)^{\sqrt{5}}} \]

and therefore

\[ 2I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{5}}} + \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{5}}}{1 + (\tan x)^{\sqrt{5}}} = \int_0^{\pi/2} \frac{1 + (\tan x)^{\sqrt{5}}}{1 + (\tan x)^{\sqrt{5}}} \]

Hence the answer above.