Cal Poly Department of Mathematics

Puzzle of the Week

April 16-22, 2010

We all know that the series $\sum_{n=1}^{\infty} \sin n$ diverges (the terms fail to approach zero). However the *partial sums* $s_N = \sum_{n=1}^{N} \sin n$ remain bounded. What is the least upper bound for the set $\{s_1, s_2, s_3, \ldots\}$?

Either an exact answer or a decimal approximation correct to 6 places will do. You may assume that $\{\sin(k) \mid k = 1, 2, 3, \ldots\}$ is dense in $[-1, 1]$.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: The least upper bound is $\dfrac{\sin 1 + 2\sin\frac{1}{2}}{2 - 2\cos 1} \approx 1.95816$.

Using the identity $e^{ix} = \cos x + i \sin x$ and geometric series we get:

$$\begin{aligned}
s_N &= \sum_{k=0}^{N} \sin k \\
&= Im\left(\sum_{k=0}^{N} e^{ik}\right) \\
&= Im\left(\frac{1 - e^{i(N+1)}}{1 - e^{i}}\right) \\
&= \frac{1}{|1 - e^{i}|^2} Im\left((1 - e^{i(N+1)})(1 - e^{-i})\right) \\
&= \frac{1}{|1 - e^{i}|^2} Im\left(1 - e^{-i} + e^{iN} - e^{i(N+1)}\right) \\
&= \frac{\sin 1 + \sin N - \sin(N+1)}{2 - 2\cos 1}
\end{aligned}$$

Now the maximum of the function $f(x) = \sin x - \sin(x+1)$ will occur at a zero of $f'(x) = \cos x - \cos(x+1)$. By symmetry these zeroes occur at $x = -\frac{1}{2} + k\pi$ for arbitrary integer k and the maximums for f correspond to the odd values for k. Plugging one into the expression above and doing a little trigonometry gives the solution.

We haven't verified that this is actually the *least* upper bound, but assuming that $\{\sin k \mid k = 1, 2, 3, \ldots\}$ is dense in $[-1, 1]$ we see that the expression above can be made arbitrarily close the solution, hence this really is the infinum.

As an aside we'll note that the fact that the partial sums of the series $\sum \sin n$ are bounded allows one to use the *Dirichlet Test* to prove that the series $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ is convergent.