Cal Poly Department of Mathematics

Puzzle of the Week

April 9-15, 2010

20 prisoners are made to stand in a line, all facing forward and none allowed to look behind himself. The evil warden places a hat on each prisoner and all the hats are either red or black. Beginning with the last person in line (who can see the 19 people ahead of him) and moving person by person to the front, the warden will ask each what color hat he has on his own head. Those who are wrong will be executed.

Devise a strategy for the prisoners which will guarantee the safety of as many as possible.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

<u>*Solution:*</u> The prisoners can guarantee the safety of 19 out of the 20(!).

This is a classic puzzle that I first heard from Jeff Liese. Here is one way to do it. The first person to speak can see 19 hats in front of him. He will say "Black" if there are an odd number of reds in front of him and "Red" otherwise. Now the next person in line can see exactly what the situation is: he can see 18 hats in front of him and knows how many of those are red. For example if the first person said "Black" (meaning the front 19 had an odd number of reds) and the second person sees an odd number of reds in front of him, then he must be wearing a black hat, but if there are an even number of reds in front of him then he will know he is wearing red. Each subsequent person simply needs to listen carefully to each person and they will be able to deduce what color hat they are wearing. Unfortunately the first person to speak can only hope that his coded message actually gave the correct answer.

Here's an interesting twist on the problem: Let's say there are now a countably infinite number of prisoners (imagine each standing above a natural number on the x-axis and facing in the positive x-direction). How many of these prisoners can be saved? Well, according to the Axiom of Choice you can save all but finitely many! To see this partition the set of all countable sequences of red and black hats by saying two are equivalent if they agree in all but finitely many places. Now the prisoners simply have to collectively agree on one representative from each equivalence class (here is where the Axiom of Choice is used). Then, when in line, each prisoner will be able to look ahead and determine which equivalence class they are in. They remember their collectively agreed upon representative and state "Red" or "Black" according to the sequence. Since this sequence agrees with the actual sequence in all but finitely many places they will eventually be correct, after only a finite number possibly incorrect answers!