Define a sequence by $a_1 = 1$ and $a_n = \cos(\tan^{-1}(a_{n-1}))$ for $n > 1$. Find

$$\lim_{n \to \infty} a_n$$

_Solutions should be submitted to Morgan Sherman:
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before next Friday. Those with correct and complete solutions will have their names listed in next week’s email announcement. Anybody is welcome to make a submission._

_Solution:_ The limit is $\sqrt{\frac{\sqrt{5} - 1}{2}}$.

Note that $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$. Now if one knows the limit exists then setting $a = \lim_{n \to \infty} a_n$ we get

$$a = \frac{1}{\sqrt{1 + a^2}}$$

from which the above solution follows. So one approach is to simply prove the limit exists.

Another approach is to recognize that $a_n = \frac{f_n}{f_{n+1}}$ where $f_n$ is the $n$th Fibonacci number. This can be quickly proved by induction:

$$a_n = \frac{1}{\sqrt{1 + a_{n-1}^2}} = \frac{1}{\sqrt{1 + \frac{f_{n-1}}{f_n}}} = \frac{\sqrt{f_n}}{\sqrt{f_n + f_{n-1}}} = \frac{f_n}{f_{n+1}}$$

Then the famous limit $f_{n+1}/f_n \to \frac{\sqrt{5} + 1}{2}$ recovers the solution above (after a little algebra).