Cal Poly Department of Mathematics

Puzzle of the Week

Feb 19-25, 2010

Define a sequence by $a_1 = 1$ and $a_n = \cos(\tan^{-1}(a_{n-1}))$ for $n > 1$. Find

$$\lim_{n\to\infty} a_n$$

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: The limit is $\sqrt{\dfrac{\sqrt{5}-1}{2}}$.

Note that $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$. Now if one knows the limit exists then setting $a = \lim_{n\to\infty} a_n$ we get

$$a = \frac{1}{\sqrt{1+a^2}}$$

from which the above solution follows. So one approach is to simply prove the limit exists.

Another approach is to recognize that $a_n = \sqrt{\frac{f_n}{f_{n+1}}}$ where f_n is the nth Fibonacci number. This can be quickly proved by induction:

$$a_n = \frac{1}{\sqrt{1+a_{n-1}^2}} = \frac{1}{\sqrt{1+\frac{f_{n-1}}{f_n}}} = \frac{\sqrt{f_n}}{\sqrt{f_n + f_{n-1}}} = \sqrt{\frac{f_n}{f_{n+1}}}$$

Then the famous limit $f_{n+1}/f_n \to \frac{\sqrt{5}+1}{2}$ recovers the solution above (after a little algebra).