Cal Poly Department of Mathematics

Puzzle of the Week

January 29 - Feb 4, 2010

This is a follow up to last week's puzzle:

A pasta dish contains n strands of spaghetti. A diner randomly picks two ends of strands (possibly of the same strand) and glues them together (with edible paste). She then repeats doing this until there are no more free ends to glue together. What is the expected number of single strand loops she will end up with?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution:

The expected number is $\dfrac{n}{2n-1}$.

Here is a slick proof, which I recieved from Lawrence Sze and Jason Estes: For any particular strand the likelihood of it becoming a single loop is $\frac{1}{2n-1}$. Summing over the n strands gives the value above.

[We've used the fact that the expected value is linear as a function on random variables. Here the random variables are X_i where $X_i = 1$ if the ith strand becomes a loop, and $X_i = 0$ if it does not. Then $E(X_i) = 1/(2n-1)$ and what we want is $E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n) = n/(2n-1)$.]

Lawrence also provides the formula for the expected number of loops of length k:

$$\frac{2^{2k-1}\binom{2n-2k}{n-k}}{k\binom{2n}{n}}$$