Cal Poly Department of Mathematics

Puzzle of the Week

January 15 - 21, 2010

Estelle Basor discovered this remarkable sum in the course of her research involving Brownian Motion. However you won't need an advanced degree to solve it as Kent Morrison has told me he has found two elementary approaches:

Let l be a non-zero integer. Find, with justification, the value of

$$\sum_{n=1}^{\infty}\left(\frac{\sin^2\frac{\pi l}{2^n}}{\frac{\pi l}{2^n}}\right)^2$$

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: If you have seen this sum (or one very similar to it) before I would appreciate it if you would please let me know. As far as I know the first person to see it was Estelle Basor.

The sum converges to 1. The only ways I have seen to go about this involve using trigonometric identities to find relations among the terms of the series and ultimately show the series telescopes. Here is one method. Using the identities

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin(2\alpha), \quad \sin^2\alpha = 1-\cos^2\alpha$$

we find:

$$\begin{aligned}
\sum_{n=1}^{N}\left(\frac{\sin^2\frac{\pi l}{2^n}}{\frac{\pi l}{2^n}}\right)^2 &= \sum_{n=1}^{N}\frac{\sin^2\frac{\pi l}{2^n}\left(1-\cos^2\frac{\pi l}{2^n}\right)}{\frac{(\pi l)^2}{4^n}} \\
&= \sum_{n=1}^{N}\frac{\sin^2\frac{\pi l}{2^n}-\left(\sin\frac{\pi l}{2^n}\cos\frac{\pi l}{2^n}\right)^2}{\frac{(\pi l)^2}{4^n}} \\
&= \sum_{n=1}^{N}\frac{\sin^2\frac{\pi l}{2^n}-\frac{1}{4}\sin^2\frac{\pi l}{2^{n-1}}}{\frac{(\pi l)^2}{4^n}} \\
&= \sum_{n=1}^{N}\left(\left(\frac{\sin\frac{\pi l}{2^n}}{\frac{\pi l}{2^n}}\right)^2-\left(\frac{\sin\frac{\pi l}{2^{n-1}}}{\frac{\pi l}{2^{n-1}}}\right)^2\right) \\
&\text{(telescopes)} \\
&= \left(\frac{\sin\frac{\pi l}{2^N}}{\frac{\pi l}{2^N}}\right)^2-\left(\frac{\sin\pi l}{\pi l}\right)^2
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty}\left(\frac{\sin^2\frac{\pi l}{2^n}}{\frac{\pi l}{2^n}}\right)^2 = \lim_{N\to\infty}\left(\frac{\sin\frac{\pi l}{2^N}}{\frac{\pi l}{2^N}}\right)^2-\left(\frac{\sin\pi l}{\pi l}\right)^2 = 1-\left(\frac{\sin\pi l}{\pi l}\right)^2$$

When l is a non-zero integer the second term vanishes and we get the answer above.