Cal Poly Department of Mathematics

Puzzle of the Week

January 8 - 14, 2010

Of all the ways to write 2010 as a sum of positive integers what is the maximal *product* of those integers?

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: The answer is 3^{670}.

There were a few different approaches taken in the various solutions I received, including: a computer assisted search (Jeremy Kun); solving the problem over $\mathbb{R}$ using Lagrange multipliers (Tom O'Neil and Jason Estes); solving the problem with arbitrary positive integer sum using indcution (Kent Morrison).

Here is the solution I came up with (as did Allan Boone): Let $a_1 + \ldots + a_n = 2010$ be such that $a_1 \cdot \ldots \cdot a_n$ is maximal. Note that for each i, $a_i \geq 2$. Furthermore note that if $a_i \geq 5$ for any i then replacing a_i with $2 + (a_i - 2)$ will give a larger product since

$$2 \times (a_i - 2) = a_i + (a_i - 4) \geq a_i + 1.$$

Furthermore replacing any $a_i = 4$ with $2 + 2$ does not change the product. Therefore for every i we have either $a_i = 2$ or $a_i = 3$. Finally, since $3+3 = 2+2+2$ but $3 \times 3 > 2 \times 2 \times 2$ we see that we can have at most two 2's. Since 3 divides evenly into 2010 the maximal product will come from

$$\underbrace{3 + 3 + \ldots + 3}_{670} = 2010$$

Hence the answer above.