Cal Poly Department of Mathematics

Puzzle of the Week

October 30 - Nov 5, 2009

Sally tightens a belt snug around the equator of the Earth. She then loosens it one notch, adding a total of one meter to the length of the belt, grabs the belt at one point and lifts it straight up from the surface of the Earth until it has no more slack. How high has Sally lifted the belt?

Note: Take the Earth to be a sphere with radius 6400 km.

Hint: Approximate answers welcome!

Solutions should be submitted to Morgan Sherman:

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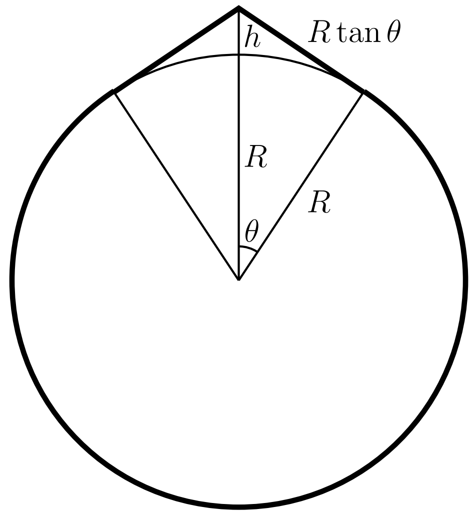
before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: This was the inaugural problem for IBM's "Ponder This" – a challenging monthly problem open to everybody (see IBM's website for more information). Let R denote the radius of the Earth and ϵ the length added to the belt. Let h denote the height Sally is holding the belt. Consider the figure on the following page and notice the right triangle with hypothenuse $h + R$:



The belt has total length $2\pi R + \epsilon$. Of this $(2\pi - 2\theta)R$ is on the surface of the Earth, while $2R\tan\theta$ is above ground. Thus $2\pi R + \epsilon = (2\pi - 2\theta)R + 2R\tan\theta$ which simplifies to

$$\tan\theta = \theta + \frac{\epsilon}{2R}. \tag{1}$$

With $R = 6.4 \times 10^6$ and $\epsilon = 1$ we can numerically solve for θ (for example the FindRoot command of Mathematica). Having done so, to solve for h one could use $\cos\theta = \frac{R}{h+R}$ which leads to

$$h = R\left(\sec\theta - 1\right). \tag{2}$$

Using Mathematica I found $\theta \approx 0.00612$ radians and $h \approx 122$ meters(!).

Alternatively, if one would like a more direct formula for h we could use Taylor series to approximate our answer, keeping in mind that we expect θ to be a small quanitity. In equations (1) and (2) we substitute the small angle approximations

$$\tan\theta \approx \theta + \frac{1}{3}\theta^3, \quad \sec\theta \approx 1 + \frac{1}{2}\theta^2.$$

Eliminating θ and solving for h leads to

$$h \approx \frac{R}{2}\theta^2 \approx \frac{R}{2}\left(\sqrt[3]{\frac{3\epsilon}{2R}}\right)^2$$

or

$$h \approx \left(\sqrt[3]{\frac{9R}{32}}\right) \cdot \epsilon^{2/3}$$

Using this formula we get the same answer, to within a tenth of a centimeter.

Perhaps the most interesting part of this problem is the uninuitive result: providing just a small amount of slack leads to a surprisingly large height. For another example consider this extreme case: if Sally were to provide just one inch of slack ($\epsilon = 1in = 0.0254m$) she would need to lift the belt nearly 35 feet into the air!