Cal Poly Department of Mathematics

Puzzle of the Week

October 23 - 29, 2009

Set $P(n) = \prod_{k=1}^{n} k^k = 1^1 \cdot 2^2 \cdot 3^3 \cdot \ldots \cdot n^n$. Define $E(n)$ to be the largest positive integer k for which 5^k divides $P(n)$. Find a closed-form formula for $E(5^m)$ and evaluate

$$\lim_{m\to\infty} \frac{E(5^m)}{5^{2m}}$$

Solutions should be submitted to Morgan Sherman:

Dept. of Mathematics, Cal Poly
Email: sherman1 -AT- calpoly.edu
Office: bldg 25 room 310

before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: This problem appeared on the 1981 Putnam exam. Let $n \le 5^m$ and note that each factor of 5 dividing n we get a contribution of n to $E(5^m)$. Each mulitple of 5 has at least one factor of 5, while multiples of 25 have a second, multiples of 125 a third, etc. In total we get a sum contribution of

$$\begin{aligned}
E(5^m) &= (5+10+15+\ldots+5^m)+ \\
&\quad +(25+50+75+\ldots+5^m)+ \\
&\quad +(125+250+375+\ldots+5^m)+ \\
&\quad \ldots+5^m \\
&= 5(1+2+3+\cdots+5^{m-1})+ \\
&\quad +5^2(1+2+3+\ldots+5^{m-2})+ \\
&\quad +5^3(1+2+3+\ldots+5^{m-3})+ \\
&\quad +\ldots+5^m(1) \\
&= \textstyle\sum_{k=1}^m 5^k(1+2+3+\ldots+5^{m-k}) \\
&= \textstyle\sum_{k=1}^m 5^k \frac{1}{2}(5^{m-k})(5^{m-k}+1) \\
&= \tfrac{1}{2}5^m \left(\textstyle\sum_{k=1}^m (5^{m-k}+1)\right) \\
&= \tfrac{1}{2}5^m \left(\tfrac{5^m-1}{5-1}+m\right) \\
&= \tfrac{1}{8}5^m\left(5^m+4m-1\right)
\end{aligned}$$

In particular the limit will be $\frac{1}{8}$.