

Cal Poly Department of Mathematics

Puzzle of the Week

October 23 - 29, 2009

Set $P(n) = \prod_{k=1}^n k^k = 1^1 \cdot 2^2 \cdot 3^3 \cdot \dots \cdot n^n$. Define $E(n)$ to be the largest positive integer k for which 5^k divides $P(n)$. Find a closed-form formula for $E(5^m)$ and evaluate

$$\lim_{m \rightarrow \infty} \frac{E(5^m)}{5^{2m}}$$

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: This problem appeared on the 1981 Putnam exam. Let $n \leq 5^m$ and note that each factor of 5 dividing n we get a contribution of n to $E(5^m)$. Each multiple of 5 has at least one factor of 5, while multiples of 25 have a second, multiples of 125 a third, etc. In total we get a sum contribution of

$$\begin{aligned} E(5^m) &= (5 + 10 + 15 + \dots + 5^m) + \\ &\quad + (25 + 50 + 75 + \dots + 5^m) + \\ &\quad + (125 + 250 + 375 + \dots + 5^m) + \\ &\quad \dots + 5^m \\ &= 5(1 + 2 + 3 + \dots + 5^{m-1}) + \\ &\quad + 5^2(1 + 2 + 3 + \dots + 5^{m-2}) + \\ &\quad + 5^3(1 + 2 + 3 + \dots + 5^{m-3}) + \\ &\quad \dots + 5^m(1) \\ &= \sum_{k=1}^m 5^k(1 + 2 + 3 + \dots + 5^{m-k}) \\ &= \sum_{k=1}^m 5^k \frac{1}{2}(5^{m-k} + 5^{m-k} + 1) \\ &= \frac{1}{2} 5^m \left(\sum_{k=1}^m (5^{m-k} + 1) \right) \\ &= \frac{1}{2} 5^m \left(\frac{5^m - 1}{5 - 1} + m \right) \\ &= \frac{1}{8} 5^m (5^m + 4m - 1) \end{aligned}$$

In particular the limit will be $\frac{1}{8}$.