Show that if an integer divides any member of the Fibonacci sequence then it divides infinitely many of them.

Note: Recall that the Fibonacci sequence \( \{f_1, f_2, f_3, \ldots \} \) is defined recursively by \( f_1 = 1, f_2 = 1, \) and for \( n > 2, \ f_n = f_{n-1} + f_{n-2}. \)

Solutions should be submitted to Morgan Sherman:

\[ \text{Dept. of Mathematics, Cal Poly} \]
\[ \text{Email: sherman1 \ AT \ calpoly.edu} \]
\[ \text{Office: bldg 25 room 310} \]

before next Friday. Those with correct and complete solutions will have their names listed in next week’s email announcement. Anybody is welcome to make a submission.

Solution: I found this problem in Larson’s book “Problem-solving through problems” (a source for more than one of this quarter’s puzzles!). People used a number of different formulas involving the Fibonacci numbers to establish this result. Most of them were either special cases of, or generalizations to \( f_{n+m} = f_{n-1}f_m + f_n f_{m+1} \) which I’m told is known as Hornsberger’s identity, but Google could not corroborate this for me. Jeff Liese provided a combinatorial proof of a different formula: \( f_n^2 + 2f_n f_{n-1} = f_{2n+1}. \) Either of these will help establish the proof.

Here is the way I went about it, using a little modular arithmetic, essentially deriving a weak form of the first formula above. Suppose \( f_k \equiv 0 \pmod{n} \). Then the next few terms in the sequence, following \( f_k \), will be, modulo \( n \):

\[
\begin{align*}
  f_k & \equiv 0 \\
  f_{k+1} & = f_{k+1} \\
  f_{k+2} & = f_k + f_{k+1} \equiv f_{k+1} \\
  f_{k+3} & \equiv f_{k+1} + f_{k+1} = f_{k+1}(1+1) = f_{k+1} \cdot f_3 \\
  f_{k+4} & \equiv f_{k+1} \cdot (1+2) = f_{k+1} \cdot f_4 \\
  \vdots \\
  f_{2k} & \equiv f_{k+1} \cdot f_k \equiv 0
\end{align*}
\]

Therefore \( n \) will also divide \( f_{2k} \) and similarly \( f_{tk} \) for every \( t \geq 1 \).