

Cal Poly Department of Mathematics

Puzzle of the Week

October 16 - 22, 2009

Show that If an integer divides any member of the Fibonacci sequence then it divides infinitely many of them.

Note: Recall that the Fibonacci sequence $\{f_1, f_2, f_3, \dots\}$ is defined recursively by $f_1 = 1$, $f_2 = 1$, and for $n > 2$, $f_n = f_{n-1} + f_{n-2}$.

Solutions should be submitted to Morgan Sherman:

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before next Friday. Those with correct and complete solutions will have their names listed in next week's email announcement. Anybody is welcome to make a submission.

Solution: I found this problem in Larson's book "Problem-solving through problems" (a source for more than one of this quarter's puzzles!). People used a number of different formulas involving the Fibonacci numbers to establish this result. Most of them were either special cases of, or generalizations to $f_{n+m} = f_{n-1}f_m + f_n f_{m+1}$ which I'm told is known as Hornsberger's identity, but Google could not corroborate this for me. Jeff Liese provided a combinatorial proof of a different formula: $f_n^2 + 2f_n f_{n-1} = f_{2n+1}$. Either of these will help establish the proof.

Here is the way I went about it, using a little modular arithmetic, essentially deriving a weak form of the first formula above. Suppose $f_k \equiv 0 \pmod{n}$. Then the next few terms in the sequence, following f_k , will be, modulo n :

$$\begin{aligned} f_k &\equiv 0 \\ f_{k+1} &= f_{k+1} \\ f_{k+2} &= f_k + f_{k+1} \equiv f_{k+1} \\ f_{k+3} &\equiv f_{k+1} + f_{k+1} = f_{k+1}(1+1) = f_{k+1} \cdot f_3 \\ f_{k+4} &\equiv f_{k+1} \cdot (1+2) = f_{k+1} \cdot f_4 \\ &\vdots \\ f_{2k} &\equiv f_{k+1} \cdot f_k \equiv 0 \end{aligned}$$

Therefore n will also divide f_{2k} and similarly f_{tk} for every $t \geq 1$.